What is Multivariable Calculus?

Limits
Derivatives
Integrals

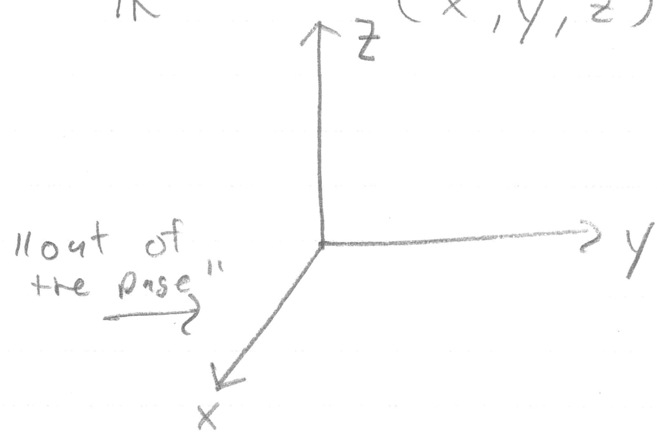
$f(x,y) = x^2 + 2y - \sqrt{xy}$

Two or more inputs $[x,y]$

Gives one output. $[f(x,y)]$
[Scalar-valued function]

- Work in 3D, you are used to 2D xy-plane. $\mathbb{R}^2$ (x,y)

We'll go to $\mathbb{R}^3$ (x,y,z)



- Vectors and their properties.
- Lines, Planes in 3D
- Calculus: Limits, Derivatives, What's different in higher dimensions

- Application/Usage of Derivatives
  - ↳ Optimatization.
  - ↳ Taylor approximations

- Vectors & Calculus

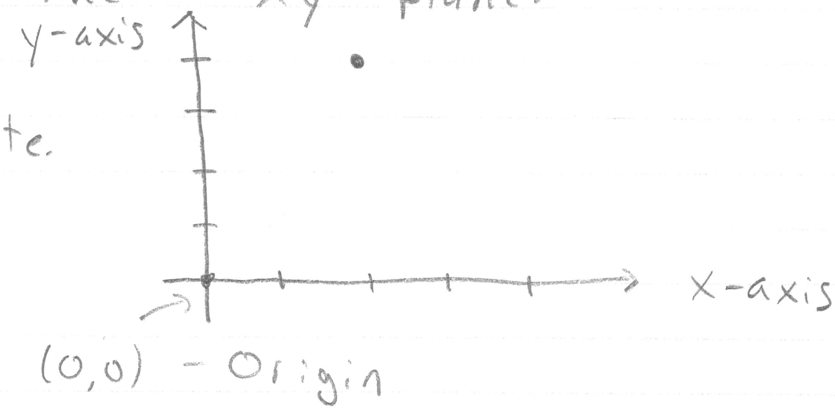
§1.1 Vectors in 2D & 3D Space

We're used to 2D, take a point (x, y) in the xy-plane.

$(2, 4)$

x-coordinate y-coordinate

$\mathbb{R}^2$



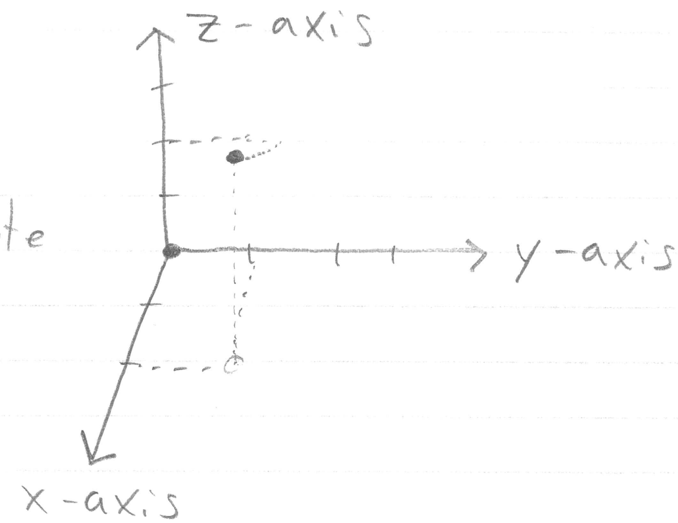
We add another axis out of/into the page

So a point is (x, y, z)

z-coordinate

$(2, 1, 2)$

$\mathbb{R}^3$



$(0, 0, 0)$ is the origin still.

We're used to working with single numbers, a <u>scalar</u>

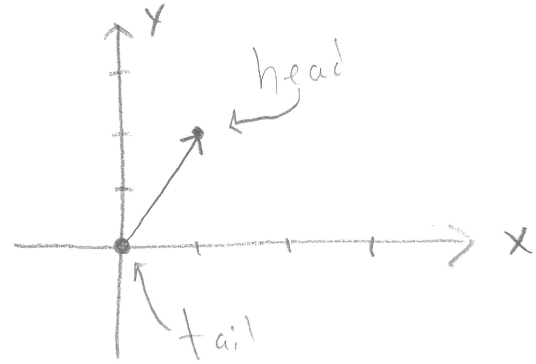
Vectors

↳ Directed line segment beginning at the origin, it has <u>magnitude</u> & <u>direction</u>.

2D Vectors

$\vec{a} = \langle 1, 2 \rangle$

↳ Bold in text.



Notice the point (1,2) is related to the vector $\vec{a}$, where the vector ends.

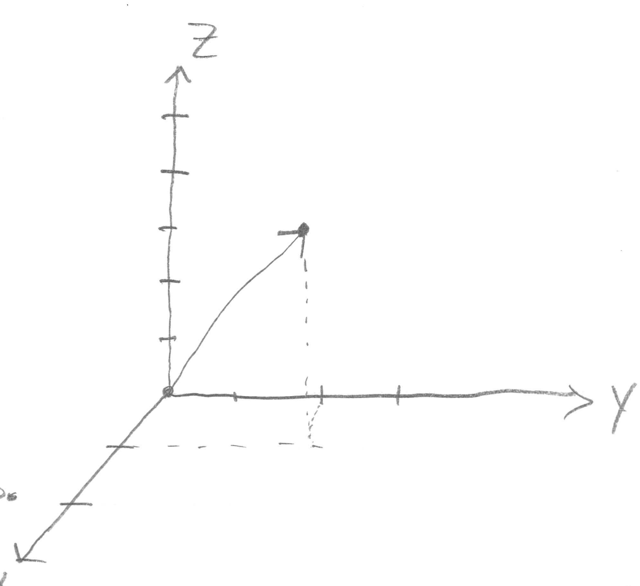
Notation:

I use Angle Brackets $\langle , \rangle$'s to denote vectors! (,)'s are points to me!. However, the textbook & webassign use (,)'s for both and (usually) make the context clear.

3D Vectors

$\vec{b} = \langle 1, 2, 3 \rangle$

Instead of x, y, z coordinates, we say x, y, z components.



Vector Properties & Operations

Some Properties:

0. The zero vector $\vec{0} = \langle 0, 0 \rangle$ or $\vec{0} = \langle 0, 0, 0 \rangle$

1. Two vectors are equal if and only if each component is equal. Say $\vec{a} = \langle a_1, a_2, a_3 \rangle$ & $\vec{b} = \langle b_1, b_2, b_3 \rangle$
$a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$.

2. Adding Vectors
$$\vec{a} + \vec{b} = \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle$$
$$= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

Ex 1| Let $\vec{u} = \langle 2, 1, 3 \rangle$ and $\vec{v} = \langle -3, 2, 0 \rangle$, find $\vec{u} + \vec{v}$.

$$\vec{u} + \vec{v} = \langle 2 + -3, 1 + 2, 3 + 0 \rangle = \langle -1, 3, 3 \rangle$$

3. Scaling Vectors
$$\alpha \langle a_1, a_2, a_3 \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$$

Ex 2| Let $\vec{u} = \langle 2, 1, 3 \rangle$, what's $2\vec{u}$?

$$2 \langle 2, 1, 3 \rangle = \langle 2 \cdot 2, 2 \cdot 1, 2 \cdot 3 \rangle = \langle 4, 2, 6 \rangle.$$

4. Subtracting Vectors:
$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Ex 3| $\vec{u} - \vec{v} = \langle 2 - -3, 1 - 2, 3 - 0 \rangle$
$$= \langle 5, -1, 3 \rangle$$

Properties Cont.

5. $(\alpha\beta)\langle a_1, a_2, a_3\rangle = \alpha(\beta\langle a_1, a_2, a_3\rangle)$
Associativity.

6. $(\alpha+\beta)\langle a_1, a_2, a_3\rangle =$

$$\alpha\langle a_1, a_2, a_3\rangle + \beta\langle a_1, a_2, a_3\rangle$$

(Distribativity)

7. $\alpha(\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle) =$
$\alpha\langle a_1, a_2, a_3\rangle + \alpha\langle b_1, b_2, b_3\rangle$
(Distribativity)

8. $\alpha\vec{0} = \alpha\langle 0, 0, 0\rangle = \langle 0, 0, 0\rangle = \vec{0}$.

9. $0\cdot\langle a_1, a_2, a_3\rangle = \langle 0, 0, 0\rangle = 0$.

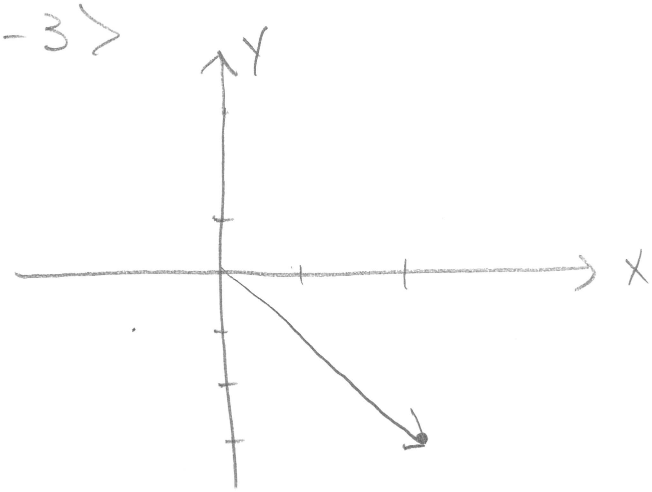
10. $1\cdot\langle a_1, a_2, a_3\rangle = \langle a_1, a_2, a_3\rangle$

All of these work in 2D as well!

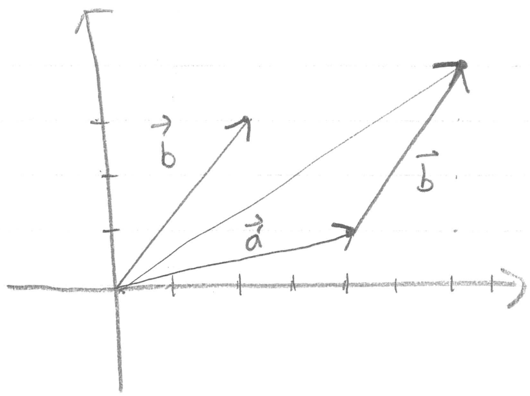
Two ways to think of a vector:
Algebraically $\langle 2, -3\rangle$

Or geometrically:

an "arrow"



Adding Geometrically



$\vec{a} = \langle 4, 1 \rangle$
$\vec{b} = \langle 2, 3 \rangle$

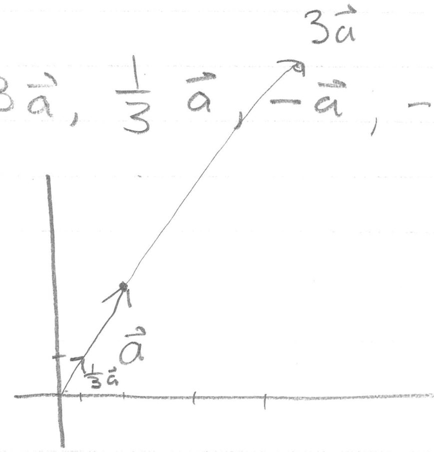
$\vec{a} + \vec{b} = \langle 6, 4 \rangle$.

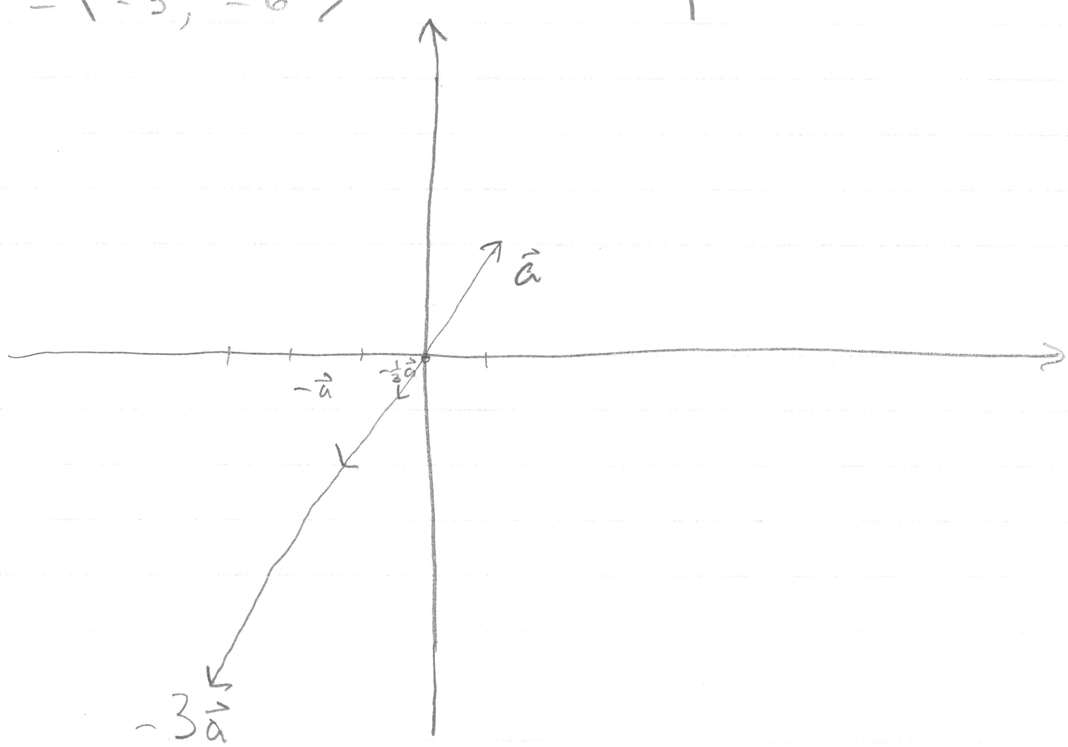
"Put tail on head"

Ex 3) Let $\vec{a} = \langle 1, 2 \rangle$

Find and sketch $3\vec{a}$, $\frac{1}{3}\vec{a}$, $-\vec{a}$, $-\frac{1}{3}\vec{a}$, $-3\vec{a}$

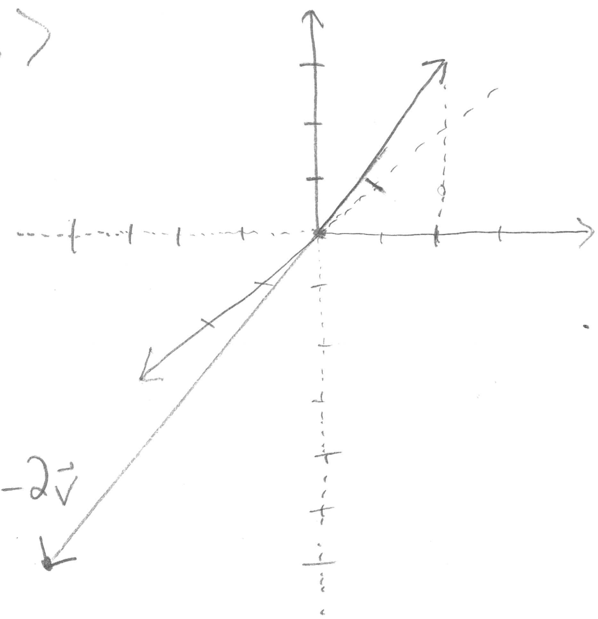
$3\vec{a} = \langle 3, 6 \rangle$
$\frac{1}{3}\vec{a} = \langle \frac{1}{3}, \frac{2}{3} \rangle$
$-\vec{a} = \langle -1, -2 \rangle$
$-\frac{1}{3}\vec{a} = \langle -\frac{1}{3}, -\frac{2}{3} \rangle$
$-3\vec{a} = \langle -3, -6 \rangle$





Ex 4 | Let $\vec{v} = \langle -1, 2, 3 \rangle$

$-2\vec{v} = \langle 2, -4, -6 \rangle$



Ex 5 | Let $\vec{a}, \vec{b}$ be 2 (different nonzero vectors)

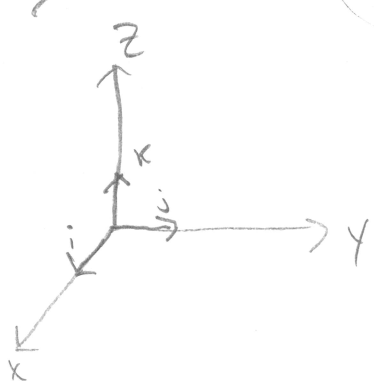
Are $3\vec{a} - \vec{b}$ & $\vec{a} - \frac{1}{3}\vec{b}$ parallel?

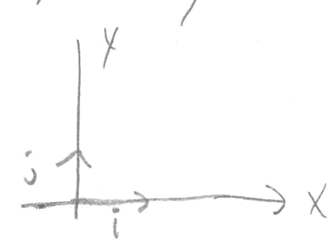
Yes, since $\frac{1}{3}(3\vec{a} - \vec{b}) = \vec{a} - \frac{1}{3}\vec{b}$

The Standard Basis Vectors $\left(\hat{i}, \hat{i}\right.$, $\hat{j}$, $\hat{j}$, $\hat{k}$, $\hat{k}$ $\left.\right)$

$i = \langle 1, 0, 0 \rangle$
$j = \langle 0, 1, 0 \rangle$
$k = \langle 0, 0, 1 \rangle$

or $i = \langle 1, 0 \rangle$
$j = \langle 0, 1 \rangle$





Ex 6 a.) Express $\vec{v} = \langle -6, \pi, 9e \rangle$ in the standard basis:

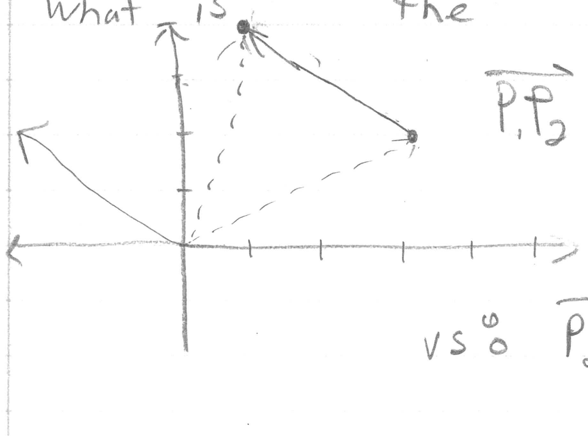
$$\vec{v} = -6\hat{i} + \pi\hat{j} + 9e\hat{k}$$

b.) What is $\vec{w} = 3i - 2k + 7j$
$$= \langle 3, 7, -2 \rangle$$

$$\boxed{\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}}$$

★ <u>The Vector Joining Two Points</u>

Ex 7 Take $P_1 = (3,2)$ & $P_2 = (1,4)$
What is the vector joining them?



$$\overrightarrow{P_1P_2} = \langle 1-3, 4-2 \rangle$$
$$= \langle -2, 2 \rangle$$

Recall: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$

vs: $\overrightarrow{P_2P_1} = \langle 3-1, 2-4 \rangle$
$$= \langle 2, -2 \rangle.$$

★ If the point P_1 has coordinates (x_1, y_1, z_1) and P_2 has coordinates (x_2, y_2, z_2), then the vector $\overrightarrow{P_1P_2}$ from the tip of P_1 to the tip of P_2 is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Lines in 3D

Remember $\underset{\substack{\uparrow \\ \text{slope}}}{y = mx} + \underset{\substack{\uparrow \\ \text{intercept (starting point)}}}{b}$?

★ Point - Direction Form of a line

$$\vec{l} = \underset{\substack{\uparrow \\ \text{point}}}{\vec{a}} + t \underset{\substack{\uparrow \\ \text{direction}}}{\vec{v}} \quad \text{or} \quad \vec{l}(t)$$

The equation of the line l through the tip of $\vec{a}$ and pointing in the direction of $\vec{v}$ is $\vec{l}(t) = \vec{a} + t\vec{v}$, where the parameter t takes on all real values. In coordinate form, the equations are:

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$$

where $\vec{a} = \langle x_1, y_1, z_1 \rangle$ & $\vec{v} = \langle a, b, c \rangle$.
In 2D, we ignore z.

★ Ex 8) Find the equation of the line through the point $(-3, 1, -2)$ in the direction of $2i - 3j + 4k$.

$$\vec{l}(t) = \langle -3, 1, -2 \rangle + t \langle 2, -3, 4 \rangle$$

or

$$\begin{cases} x = -3 + 2t \\ y = 1 - 3t \\ z = -2 + 4t \end{cases}$$

Ex 9) What direction does $\begin{cases} x = 2 + 6t \\ y = -3(t-1) \\ z = -4t + 5 \end{cases}$ point?

Find $\underline{\vec{v} = \langle 6, -3, -4 \rangle}$

Lines <u>cont.</u>

<u>Ex 10</u> Do the lines $\vec{l}_1 = \langle 3t+1, 3t, 0 \rangle$ and $\vec{l}_2 = \langle t, -6t+3, 2t-8 \rangle$ intersect?

[2D
$y_1 = m_1 x + b_1$? $\Rightarrow y_1 = m_1 x + b_1 \overset{?}{=} m_2 x + b_2 = y_2$
$y_2 = m_2 x + b_2$ for 1 x-value (or infinite)]

$$\langle 3t_1 + 1, 3t_1, 0 \rangle = \langle t_2, -6t_2 + 3, 2t_2 - 8 \rangle$$

$$\begin{cases} 3t_1 + 1 = t_2 \\ 3t_1 = -6t_2 + 3 \\ 0 = 2t_2 - 8 \end{cases}$$

all must be true.

$2t_2 = 8 \Rightarrow t_2 = 4$

$3t_1 + 1 = 4 \quad 3t_1 = 3 \quad t_1 = 1$

$2 \rightarrow 3(1) = -6(4) + 3$

$3 \neq -21.$

The lines don't intersect.

Lines cont.

★ Point - Point Form

$P_1 = (x_1, y_1, z_1)$ & $P_2 = (x_2, y_2, z_2)$

$$\begin{cases} x = x_1 + (x_2 - x_1)\,t \\ y = y_1 + (y_2 - y_1)\,t \\ z = z_1 + (z_2 - z_1)\,t \end{cases}$$

2D

$y = mx + b$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$y = \left(\dfrac{y_2 - y_1}{x_2 - x_1}\right) x + b$

Ex 11) Find the equation of the line passing through $(1, -4, -3)$ and $(3, 1, -6)$

$$\begin{cases} x = 1 + (3 - 1)\,t = 1 + 2t \\ y = -4 + (1 - -4)\,t = -4 + 5t \\ z = -3 + (-6 - -3)\,t = -3 - 3t \end{cases}$$

or $\vec{l}(t) = \langle 1, -4, 3 \rangle + \langle 2, 5, -3 \rangle t$

$= \langle 1 + 2t,\ -4 + 5t,\ -3 - 3t \rangle$

§1.2 The (Dot) Inner Product, Length, & Distance.

We know one way to multiply vectors so far.... scaling them $2 \cdot \langle 1, -2, 3 \rangle = \langle 2, -4, 6 \rangle$

scalar times vector = vector.
↳ scalar

vector times vector = scalar

↳ Inner Product or <u>Dot</u> product.

Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} = \langle b_1, b_2, b_3 \rangle$

$$\star \; \boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

"Multiply components then add."

<u>Ex1|</u> $\vec{a} = \langle 1, -1, 2 \rangle$ & $\vec{b} = \langle 3, -4, -2 \rangle$
$\vec{a} \cdot \vec{b}$?

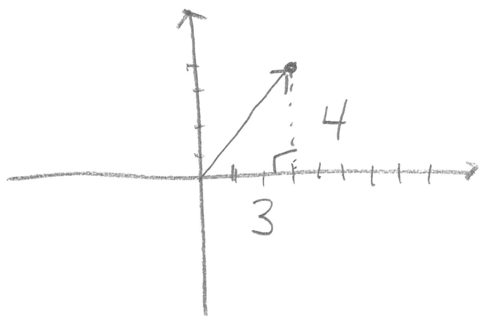
$$\vec{a} \cdot \vec{b} = 1 \cdot 3 + (-1)(-4) + 2(-2)$$
$$= 3 + 4 - 4 = 3$$

<u>Ex2|</u> If $\vec{u} = 3\hat{i} - 7\hat{k}$ & $\vec{v} = -\hat{i} + 2\hat{j} + 2\hat{k}$
What is $\vec{u} \cdot \vec{v}$?

$$\vec{u} \cdot \vec{v} = 3(-1) + 0(2) + (-7)2$$
$$= -3 + 0 - 14 = -17$$

*Works for 2D too!
[and higher]

Ex 3 | How long is the vector $\vec{a} = \langle 3, 4 \rangle$?



Use Pythagorean Theorem:

$$\sqrt{3^2 + 4^2} = 5$$
$$\sqrt{a_1^2 + a_2^2}$$

This vector has length 5.

☆ Defn: Length, Norm, Magnitude (of a vector)

Given $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{a} \cdot \vec{a}}$$

So $\|\vec{a}\| = \sqrt{3^2 + 4^2} = 5$

Properties of the Dot Product:

1. $\vec{a} \cdot \vec{a} \geq 0$
   & $\vec{a} \cdot \vec{a} = 0$ if and only if $\vec{a} = \vec{0}$.

2. $\alpha \vec{a} \cdot \vec{b} = \alpha (\vec{a} \cdot \vec{b})$
   $\vec{a} \cdot \alpha \vec{b} = \alpha (\vec{a} \cdot \vec{b})$

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
   & $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

4. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Unit Vectors & Distance

Sometimes we only need a direction.
Vectors with length 1, are <u>unit vectors</u>.

For any nonzero vector, $\frac{\vec{a}}{\|\vec{a}\|}$ is a unit vector

We have <u>normalized</u> it to have length one.

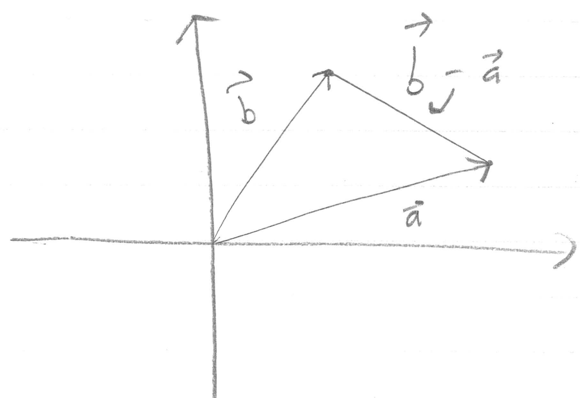
<u>Ex 4</u>| Normalize $\vec{w} = \langle -2, 1, 5 \rangle = -2\hat{i} + \hat{j} + 5\hat{k}$

$$\|\vec{w}\| = \sqrt{(-2)^2 + (1)^2 + (5)^2} = \sqrt{4+1+25} = \sqrt{30}$$

So $\frac{\vec{w}}{\|\vec{w}\|} = \left\langle \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}} \right\rangle$

<u>Distance:</u>

Distance between the tip of $\vec{a}$ & $\vec{b}$ is $\|\vec{b} - \vec{a}\|$



<u>Ex 5</u>| Find the distance between $(0,0,1)$ & $(1,0,0)$. [The tip of $\hat{k}$ & $\hat{i}$].

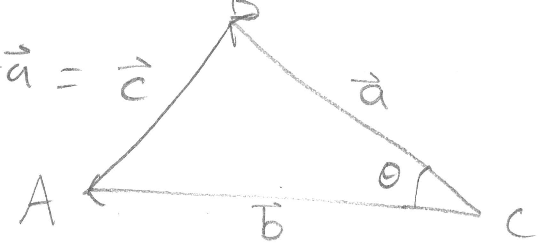
$$\langle 1-0, 0-0, 0-1 \rangle = \langle 1, 0, -1 \rangle$$

$$\|\langle 1, 0, -1 \rangle\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

Angle between vectors

Remember the Law of Cosines?

$\vec{b}-\vec{a}=\vec{c}$



$$c^2 = a^2+b^2-2ab\cos(C)$$

$$\|\vec{b}-\vec{a}\|^2 = \|\vec{a}\|^2+\|\vec{b}\|^2-2\|\vec{a}\|\|\vec{b}\|\cos(\theta)$$

$$(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) = \|\vec{a}\|^2+\|\vec{b}\|^2-2\|\vec{a}\|\cdot\|\vec{b}\|\cos(\theta)$$

$$\vec{b}\cdot\vec{b}-\vec{b}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{a}$$

$$\cancel{\|\vec{b}\|^2}-2\vec{a}\cdot\vec{b}+\cancel{\|\vec{a}\|^2} = \cancel{\|a\|^2}+\cancel{\|b\|^2}-\|\vec{a}\|\|\vec{b}\|\cos(\theta)$$

$$-2\vec{a}\cdot\vec{b} = -2\|\vec{a}\|\|\vec{b}\|\cos(\theta)$$

$$\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos(\theta)}$$

Or

$$\theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$$

☆ **Ex 6** Find the angle between $\vec{u}=-2\hat{i}+\hat{j}+5\hat{k}$ & $\vec{v}=2\hat{i}-\hat{j}+4\hat{k}$

$$\theta = \arccos\left(\frac{-2(2)+1(-1)+5(4)}{\sqrt{(-2)^2+(1)^2+(5^2)}\sqrt{(2)^2+(-1)^2+(4)^2}}\right)$$

$$= \arccos\left(\frac{-4-1+20}{\sqrt{4+1+25}\sqrt{4+1+16}}\right)$$

$$= \arccos\left(\frac{15}{\sqrt{30}\sqrt{21}}\right) \approx 0.93 \text{ rads or } 53.3^\circ$$

Using Calculator.

Useful Facts:

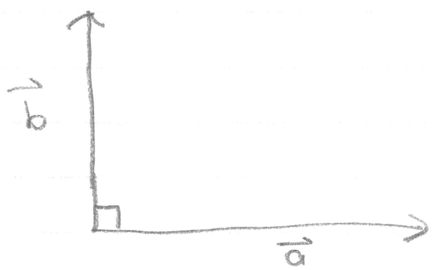
1. Useful Inequality:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos(\theta)$$

$$|\vec{a} \cdot \vec{b}| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot |\cos(\theta)|$$
$$\leq \|\vec{a}\| \, \|\vec{b}\| \cdot 1$$

$\Rightarrow$ Cauchy - Schwarz Inequality: $\boxed{|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \cdot \|\vec{b}\|}$

2. Orthogonal / Perpendicular Vectors



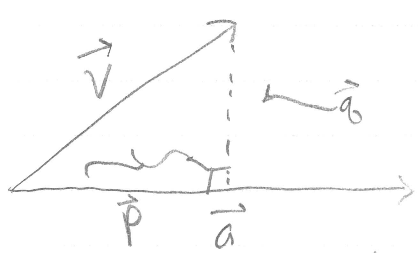
What is $\vec{a} \cdot \vec{b}$?

$$\cos(90^\circ) = \cos\left(\frac{\pi}{2}\right) = 0$$

So, $\vec{a} \cdot \vec{b} = 0$.

Vectors that are perpendicular or orthogonal, have dot product zero; if the dot product is zero, the vectors are perpendicular.

3. Orthogonal Projection (Shadow of a vector on another)



$\vec{p} = c\vec{a}$ for some c.

$$\vec{v} = c\vec{a} + \vec{q}$$
$$\vec{v} \cdot \vec{a} = c\vec{a} \cdot \vec{a} + \cancel{\vec{q} \cdot \vec{a}}$$
$$\vec{v} \cdot \vec{a} = c\|a\|^2 \quad \Rightarrow \quad c = \frac{\vec{v} \cdot \vec{a}}{\|a\|^2}$$

$$\boxed{\vec{p} = \frac{\vec{v} \cdot \vec{a}}{\|a\|^2} \cdot \vec{a}}$$

The orthogonal projection of $\vec{v}$ on $\vec{a}$ is $\vec{p}$.

Ex 7| Find the orthogonal projection of $\langle 1, 5 \rangle$ on $\langle -1, 4 \rangle$

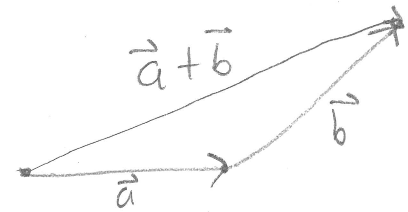
$\vec{v}$ $\qquad$ $\vec{a}$ $\qquad$ $\|\vec{a}\| = \sqrt{17}$

$$\vec{v} \cdot \vec{a} = -1 + 20 = 19$$

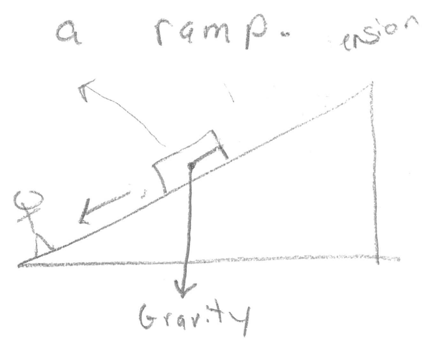
$$\vec{p} = \frac{19}{\sqrt{17}} \langle -1, 4 \rangle$$

4. Triangle Inequality

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|.$$



Ex 8| Physics! Forces are a vector.
Consider a box on a ramp.



$$\vec{F}_g = \langle 0, 0, -10 \rangle$$

$$\vec{F}_{rope} = \langle -3, 4, 1 \rangle$$

$\vec{F}_{you}$?

$$\vec{F}_g + \vec{F}_{rope} + \vec{F}_{you} = \vec{0}$$

$$-10\hat{k} - 3\hat{i} + 4\hat{j} + \hat{k} + \vec{F}_{you} = \vec{0}$$

$$\vec{F}_{you} = 3\hat{i} - 4\hat{j} + 9\hat{k}$$

§1.3 Matrices, Determinants, & the Cross Product

Matrices

A 2×2 matrix is an array of scalars

Brackets ↔ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\qquad a_{1,2}$ ← 2nd Column; ↑ 1st Row

E.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

A useful number associated with a matrix is it's determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

matrix $\qquad\qquad$ it's determinant.

Ex 1

a) $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 0 = 1$

b) $\begin{vmatrix} 2 & 4 \\ -1 & 0 \end{vmatrix} = 2\cdot 0 - 4(-1) = 4$

c) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = -2$

Matrices <u>cont.</u>

A 3×3 matrix is similar!

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

<u>Eg)</u> $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

A 3×3 determinant is more complicated....

$$\boxed{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

<u>Ex2)</u> $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 1(5\cdot 9 - 8\cdot 6) - 2(4\cdot 9 - 7\cdot 6) + 3(4\cdot 8 - 7\cdot 5)$

$= (45 - 48) - 2(36 - 42) + 3(32 - 35)$

$= -3 - 2(-6) + 3(-3)$

$= -3 + 12 - 9 = 0.$

Cross Product

Another way to multiply vectors, only in $\mathbb{R}^3$, it produces another vector!

Defn: The Cross Product (or Vector Product) Suppose $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + a_3\hat{k}$ are vectors in $\mathbb{R}^3$. The cross product of $\vec{a}$ and $\vec{b}$, denoted $\vec{a} \times \vec{b}$ is, the vector:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Ex31 Let $\vec{a} = \langle 1, 3, 4 \rangle$ & $\vec{b} = \langle 2, 7, -5 \rangle$
Find $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \hat{i}\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

$$= \hat{i}(-15 - 28) - \hat{j}(-5 - 8) + \hat{k}(7 - 6)$$

$$= -43\hat{i} + 13\hat{j} + 1\hat{k} = \langle -43, 13, 1 \rangle$$

This new vector is perpendicular to both $\vec{a}$ and $\vec{b}$, let's check:

$$\langle 1, 3, 4 \rangle \cdot \langle -43, 13, 1 \rangle = -43 + 39 + 4 = 0$$

$$\langle 2, 7, -5 \rangle \cdot \langle -43, 13, 1 \rangle = -86 + 91 - 5 = 0.$$

Properties of the Cross Product

1. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ (Order Matters!)
2. $\alpha\vec{a} \times \vec{b} = \alpha(\vec{a} \times \vec{b})$

   $\vec{a} \times (\beta\vec{b} + \gamma\vec{c}) = \beta(\vec{a} \times \vec{b}) + \gamma(\vec{a} \times \vec{c})$

   $(\alpha\vec{a} + \beta\vec{b}) \times \vec{c} = \alpha(\vec{a} \times \vec{c}) + \beta(\vec{b} \times \vec{c})$

3. $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

4. The Righthand rule - take right hand index in direction of $\vec{a}$, middle in $\vec{b}$, thumb tells you direction of $\vec{a} \times \vec{b}$.

5. $\vec{a} \times \vec{b} = 0$ if and only if $\vec{a}$ & $\vec{b}$ are parallel or $\vec{a}$ or $\vec{b}$ is zero.

   $\Rightarrow$ $\boxed{\vec{a} \times \vec{a} = 0}$

6. $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ & $\vec{b}$.

7. $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\,\|\vec{b}\| \sin(\theta) =$ the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ $(0 \leq \theta \leq \pi)$. [Triangle is $\frac{1}{2}\|\vec{a} \times \vec{b}\|$]

Ex 4] Find the area of the parallelogram spanned by $\vec{a} = \langle -3, 1, -7 \rangle$ & $\vec{b} = \langle 0, -5, -5 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} - \hat{j}\begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix}$$

$$= \langle -5 - 35, -(15 - 0), 15 - 0 \rangle = \langle -40, -15, 15 \rangle$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{(-40)^2 + (15)^2 + (15)^2} = 5\sqrt{82}$$

Relating Determinants to Geometry

2×2:

The determinant of a 2×2 matrix $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

is $a_1 b_2 - b_1 a_2$.

Imagine we have vectors $\vec{a} = \langle a_1, a_2, 0 \rangle$
$\vec{b} = \langle b_1, b_2, 0 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \hat{k}(a_1 b_2 - b_1 a_2)$$

So $\quad \| \vec{a} \times \vec{b} \| = \sqrt{(a_1 b_2 - b_1 a_2)^2} = |a_1 b_2 - b_1 a_2|$

Hence, the area of a parallelogram is the absolute value of the determinant.

3×3:

Consider $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ &
$\vec{c} = \langle c_1, c_2, c_3 \rangle$

These form a parallelopiped.

The absolute value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

is the volume of the parallelopiped.

Ex 51 Find the volume of the parallelopiped spanned by

$$\vec{a} = -1\hat{i} - 2\hat{j} - 3\hat{k}$$
$$\vec{b} = -1\hat{i} + 1\hat{j} + 2\hat{k}$$
$$\vec{c} = 2\hat{i} + 1\hat{j} + 4\hat{k}$$

$$\begin{vmatrix} -1 & -2 & -3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - -2\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + -3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= -1(4-2) + 2(-4-4) - 3(-1-2)$$
$$= -2 - 16 + 9$$
$$= -9$$

Thus the volume is 9.

Planes

Now think back to lines we had $y = mx + b$ but also $ax + by = c$ was a line. We generalize this form, unfortunately in 3D, it's a plane!

Equation of a plane in space:
The equation of a plane through (x_0, y_0, z_0) with normal vector $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

So, a point (x, y, z) is on the plane if and only if $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$.

Planes cont.

Ex 6) Find an equation of the plane through the point $(2, 4, -1)$ and with normal vector $\vec{n} = \langle 2, 3, 4 \rangle$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$
$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$
$$2x - 4 + 3y - 12 + 4z + 4 = 0$$

$$2x + 3y + 4z = 12$$

What's the x-intercept?
set $y = 0$ & $z = 0$ $\quad 2x = 12$
$x = 6.$

★ **Ex 7)** Find an equation of the plane through the points $P = (1, 3, 2)$, $Q = (3, -1, 6)$, & $R = (5, 2, 0)$.

We need the normal vector. Since these 3 points are on the plane, I'll form 2 vectors in the plane $\overrightarrow{PQ}$ & $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = \langle 3 - 1, -1 - 3, 6 - 2 \rangle = \langle 2, -4, 4 \rangle$$
$$\overrightarrow{PR} = \langle 5 - 1, 2 - 3, 0 - 2 \rangle = \langle 4, -1, -2 \rangle$$

Now I will use the cross product to product the normal vector at point P.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$8 - -4, -(-4 - 16), -2 - -16$

$$= \hat{i} \begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix} = \langle 12, 20, 14 \rangle$$

__Ex 2 cont.__ Thus $\vec{n} = \langle 12, 20, 14 \rangle$

So our plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

or

$$12x + 20y + 14z = 50.$$

In 2D, two lines were parallel when they had the same slope, m.

In 3D, two planes are parallel when they have the same normal vector, $\vec{n}$.

__Ex 8__ Find the intersection of

$$-2x + 3y + z = 1$$ and
$$2x + 3y - z = 0$$

Solve the system!

add

$$6y = 1 \qquad y = \frac{1}{6}$$

Only 1 Eqn. 2 unknowns!

$$2x + \frac{1}{2} - z = 0$$

let x be free, our parameter, so

$$2t + \frac{1}{2} - z = 0$$

$$z = 2t + \frac{1}{2}$$

Solution:

$$\begin{cases} x = t \\ y = \frac{1}{6} \\ z = 2t + \frac{1}{2} \end{cases}$$

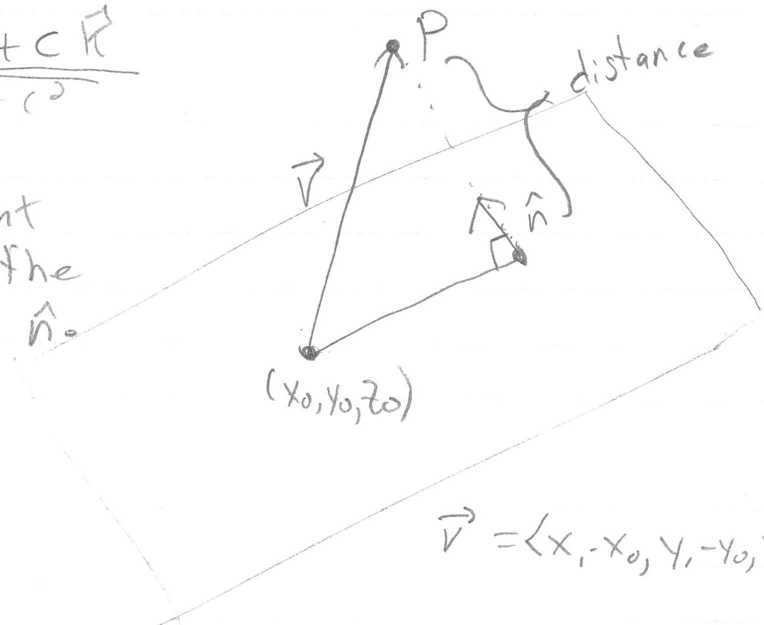
(A line)

$$\vec{l}(t) = \langle 0, \tfrac{1}{6}, \tfrac{1}{2} \rangle + t\langle 1, 0, 2 \rangle.$$

Distance from a point to a plane.
Consider the point $P = (x_1, y_1, z_1)$ and the plane $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Let's consider the unit normal vector

$$\hat{n} = \frac{A\hat{i} + B\hat{j} + C\hat{k}}{\sqrt{A^2 + B^2 + C^2}}$$



The distance we want is the length of the projection $\vec{v}$ onto $\hat{n}$.

$\vec{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

So

$$d = \left\| \frac{\hat{n} \cdot \vec{v}}{\|\hat{n}\|^2} \hat{n} \right\|$$

$$= \| \hat{n} \cdot \vec{v} \, \hat{n} \| = | \hat{n} \cdot \vec{v} |$$

$$= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}}$$

or

$$\boxed{d = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}}$$

where $D = Ax_0 + By_0 + Cz_0$

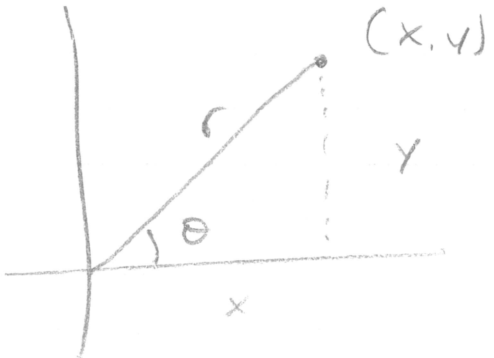
Ex 9| Find the distance between the point $(1, -2, 4)$ and the plane $3x + 2y + 6z = 5$

$$d = \frac{|3(1) + 2(-2) + 6(4) - 5|}{\sqrt{3^2 + 2^2 + 6^2}}$$

$$= \frac{|3 - 4 + 24 - 5|}{\sqrt{9 + 4 + 36}} = \frac{18}{7}$$

§ 1.4 Cylindrical & Spherical Coordinates

Remember in 2D we have our usual Rectangular coordinates (x,y), but also we had polar (r,θ)



$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

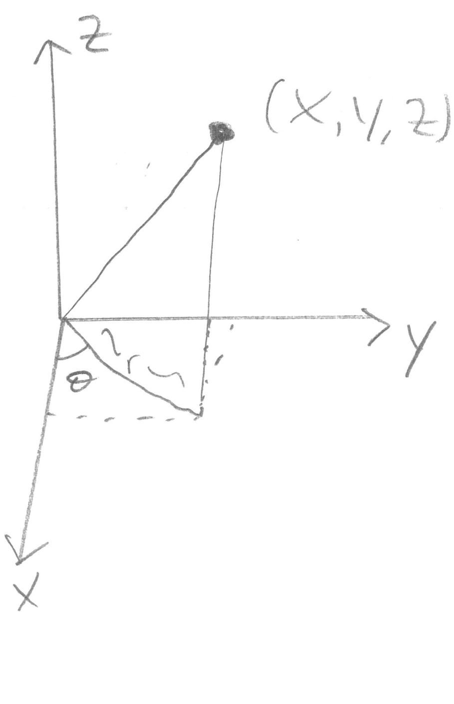
$$r = \sqrt{x^2+y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

We'll learn two more...

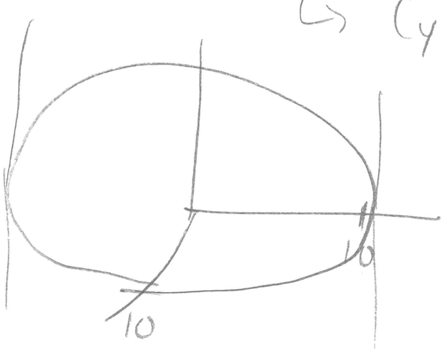
1. Cylinderical coordinates
   (It's like polar but with $z = z$)

<u>Defn:</u>
The cylinderical coordinates (r, θ, z) of a point (x,y,z) are defined by

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$



<u>Eg.)</u> $r = 10$ ↳ Cylinder with radius 10.



Cylinderical Coordinates
Cont.

Ex 1) Given the cylinderical coordinates $(2, \frac{2\pi}{3}, 1)$, find its rectangular coordinates.

(r, θ, z)

$x = 2\cos\left(\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$
$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$
$z = 1 = 1$

So $(-1, \sqrt{3}, 1)$ is the rectangular coordinates.

Ex 2) Given the rectangular coordinates $(3, -3, -7)$, find the cylinderical ones.
$x \quad y \quad z$

$r = \sqrt{3^2 + (-3)^2} = \sqrt{18}$

$\theta = \arctan\left(\frac{-3}{3}\right) = \arctan(-1) = -\frac{\pi}{4}$

$z = -7$

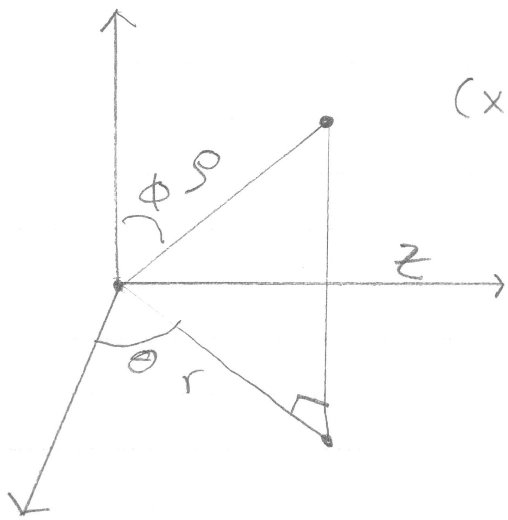
So $(\sqrt{18}, -\frac{\pi}{4}, -7)$ is a cylinderical coordinate.

Another is $(\sqrt{18}, \frac{7\pi}{4}, -7)$

there are infinitely many!

Spherical Coordinates

A better analogue from polar coordinates in 2D to 3D, would be <u>spherical</u> coordinates.



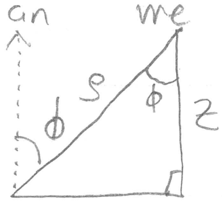
θ, r come from our usual polar coordinates

$x = r\cos\theta$ $\quad r = \sqrt{x^2+y^2}$

$y = r\sin\theta$

$\rho = \sqrt{x^2+y^2+z^2}$ (the magnitude of the relevant vector)

Can we express r, z in terms of ρ, ϕ?



Use Trig! S.O.H.

$\sin(\phi) = \dfrac{r}{\rho}$

$r = \rho\sin(\phi)$

C.A.H.

$\cos(\phi) = \dfrac{z}{\rho}$

$z = \rho\cos(\phi)$ $\longrightarrow$ $\phi = \arccos\left(\dfrac{z}{\rho}\right)$

Defn: Spherical Coordinates
The spherical coordinates (ρ, θ, ϕ) of a point (x, y, z) are defined by:

$$x = \rho\sin(\phi)\cos(\theta) \qquad \rho \geq 0$$
$$y = \rho\sin(\phi)\sin(\theta) \qquad 0 \leq \theta < 2\pi$$
$$z = \rho\cos(\phi) \qquad 0 \leq \phi \leq \pi$$

Ex3) Given the rectangular coordinates $(3, -3, -7)$ find the spherical coordinates.

$$\rho = \sqrt{3^2 + (-3)^2 + (-7)^2} = \sqrt{9+9+49} = \sqrt{67}$$

$$\theta = \arctan\left(\frac{-3}{3}\right) = -\frac{\pi}{4} \ (+2\pi) = \frac{7\pi}{4}$$

$$\phi = \arccos\left(\frac{-7}{\sqrt{67}}\right) \approx 148.8^\circ$$

So in spherical we have $\left(\sqrt{67}, \frac{7\pi}{4}, \arccos\left(\frac{-7}{\sqrt{67}}\right)\right)$.

Ex4) Convert $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ to rectangular, then to cylinderical.

$$x = 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$
$$y = 2\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$
$$z = 2\cos\left(\frac{\pi}{3}\right) = 2\,\frac{1}{2} = 1$$

So it's $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1\right)$ in rectangular.
For cylinderical we just need $r = \sqrt{x^2+y^2}$

$$r = \sqrt{\frac{3}{4} + \frac{3}{4}} = \frac{\sqrt{6}}{2} \qquad \left(\frac{\sqrt{6}}{2}, \frac{\pi}{4}, 1\right).$$

§1.5 n-Dimensional Euclidean Space

We have $\mathbb{R} = \mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ so far.
We can generalize to $\mathbb{R}^n$.
A vector might look like
$\vec{x} = \langle x_1, x_2, x_3, \dots, x_{n-1}, x_n \rangle$
n-components.
Or perhaps a point $(x_1, x_2, x_3, \dots, x_{n-1}, x_n)$

Instead of $\hat{i}, \hat{j}, \hat{k}$ in $\mathbb{R}^3$ we have
$\vec{e_1} = \langle 1, 0, 0, \dots, 0 \rangle$
$\vec{e_2} = \langle 0, 1, \dots, 0 \rangle$
$\vdots$
$\vec{e_n} = \langle 0, 0, 0, \dots, 1 \rangle$
in $\mathbb{R}^n$.

Dot Product still works similarly:
$\langle x_1, x_2, x_3, \dots, x_n \rangle \cdot \langle y_1, y_2, y_3, \dots, y_n \rangle$
$= x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n$

Norm still works:

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\vec{x} \cdot \vec{x}} = \text{Length}$$

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos(\theta)$$

still works and makes sense to think about.

Thm: Properties of Dot Product in $\mathbb{R}^n$

T3
i) $(\alpha\vec{x} + \beta\vec{y}) \cdot \vec{z} = \alpha(\vec{x} \cdot \vec{z}) + \beta(\vec{y} \cdot \vec{z})$

ii) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

iii) $\vec{x} \cdot \vec{x} \geq 0 \qquad (\|\vec{x}\| \geq 0)$

iv) $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = 0$

T4 v.) $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$

Col vi) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

__Ex 11__ Let $\vec{a} = \langle 1, 2, 3, 4 \rangle$ &
$\vec{b} = \langle -1, 0, 2, 1 \rangle$
Find $\vec{a} \cdot \vec{b}$ and check v, vi.

$\vec{a} \cdot \vec{b} = 1(-1) + 2(0) + 3(2) + 4(1) = 9$

$\|\vec{a}\| = \sqrt{1 + 4 + 9 + 16} = \sqrt{30}$
$\|\vec{b}\| = \sqrt{1 + 0 + 4 + 1} = \sqrt{6}$

$\sqrt{81} = 9 \overset{?}{\leq} \sqrt{30} \cdot \sqrt{6} = \sqrt{180}$ ✓
(≈ 13.4)

$\vec{a} + \vec{b} = \langle 0, 2, 5, 5 \rangle$

$\|\vec{a} + \vec{b}\| = \sqrt{0 + 4 + 25 + 25} = \sqrt{54}$

$\sqrt{54} \overset{?}{\leq} \sqrt{30} + \sqrt{6}$

$7.3 \leq 5.4 + 2.4 = 7.9$ ✓

Matrices

In general we can have $m \times n$ matrices
(rows) $\times$ (columns)

an array of $m \cdot n$ scalars:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$$

Basic Operations:

1. Add two matrices of the same size by adding corresponding entries.

2. Scale entire matrix by a constant. Multiply the constant into each entry.

Ex 2 a)

$$\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 6 & -7 \\ 5 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & -2+6 & 3-7 \\ -4+5 & 5+3 & 2+0 \end{bmatrix} = \begin{bmatrix} 9 & 4 & -4 \\ 1 & 8 & 2 \end{bmatrix}$$

b)

$$3\begin{bmatrix} -1 & 2 & 4 \\ 1 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 12 \\ 3 & 9 & 0 \\ 0 & 3 & -3 \end{bmatrix}$$

Multiplying Matrices

Defn: Matrix Multiplication
If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $n \times n$ matrices then the product $AB = C$ has entries

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

Or, the dot product of the i-th row of A and j-th column of B.

nth column

2nd row

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \ddots & \\ \vdots & & & \\ b_{n1} & \cdots & & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & c_{2n} \\ & & \\ c_{n1} & & c_{nn} \end{bmatrix}$$

Multiplying Matrices <u>cont</u>

Ex 3 Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

Find AB and BA.

$$AB = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1\cdot 0 + 1\cdot 2 + 0\cdot 1 & 1\cdot 1 + 1\cdot 0 + 0\cdot 0 & 1\cdot 0 + 1(-1) + 0\cdot 3 \\ 0\cdot 0 + 0\cdot 2 + 1\cdot 1 & 0\cdot 1 + 0\cdot 0 + 0\cdot 1 & 0\cdot 0 + 0(-1) + 1(3) \\ 0\cdot 0 + 1\cdot 2 + 0\cdot 1 & 0\cdot 1 + 1\cdot 0 + 0\cdot 0 & 0\cdot 0 + 1(-1) + 0(3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ 2 & 0 & -1 \end{bmatrix} = AB$$

$$BA = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 0 + 0\cdot 1 + 1\cdot 0 & 3\cdot 1 + 0\cdot 0 + 1\cdot 1 & 3\cdot 0 + 0\cdot 0 + 1\cdot 1 \\ -1\cdot 0 + 0\cdot 1 + 2\cdot 0 & -1\cdot 1 + 0\cdot 0 + 2\cdot 1 & -1\cdot 0 + 0\cdot 0 + 2\cdot 1 \\ 0\cdot 0 + 1\cdot 1 + 0\cdot 0 & 0\cdot 1 + 1\cdot 0 + 0\cdot 1 & 0\cdot 0 + 1\cdot 0 + 0\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Notice $AB \neq BA$

Properties

Property 1: In general, $AB \neq BA$ for matrices!

Property 2: We can extend matrix multiplication further to any size, however for AB to be defined the number of columns of A must be equal to the rows of B.

Property 3: An $n \times n$ matrix is said to be invertible if there exists an $n \times n$ matrix B, so that $AB = BA = I_n$

where $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & 1 & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$, the identity matrix.

[Think $7 \cdot \frac{1}{7} = 1$, 7 is invertible, with multiplicative inverse $\frac{1}{7}$.]

We denote B by A^{-1}, if A^{-1} exists it's unique. $[7 \cdot 7^{-1} = 1]$

Property 4: The determinant of a matrix A, is denoted $\det(A)$. Given A, B,

$$\det(AB) = \det(A)\det(B).$$

Property 5: An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.