§2.1 The Geometry of Real-Valued Functions

We're used to working with functions:

e.g. $f(x) = x^2 + \sin(x)$

$$f: \underset{\text{domain}}{\mathbb{R}} \to \underset{\text{Codomain}}{\mathbb{R}}$$

Generally these get shrunk to smaller sets. $A \subseteq \mathbb{R}$ and Range.

e.g $f(x) = \sqrt{x}$

$$f: \underset{\subset \mathbb{R}}{[0, \infty)} \to \underset{\subset \mathbb{R}}{[0, \infty)}$$

We extend this idea to higher dimensions...

<u>Defn</u>: Function

$$f: \mathbb{R}^n \to \mathbb{R}^m$$

A is the domain of f, $A \subseteq \mathbb{R}^n$. With Range, or $\text{Ran}(f) \subseteq \mathbb{R}^m$.

So $\vec{x} = \underbrace{\langle x_1, x_2, x_3, \ldots x_n \rangle}_{\text{n-tuple}} \in A$ or $x = \underbrace{(x_1, x_2, \ldots x_n)}_{\text{n-tupe.}} \in A$

Input an n-tuple, output an m-tuple $f(\vec{x})$. $\vec{x} \mapsto f(\vec{x})$

If $m=1$, we call f a real-valued function or scalar-valued function.

If $m>1$, we call f a vector-valued function.

If $n>1$ such functions are called functions of several variables.

E.g. 1] $f(x,y,z) = \dfrac{1}{x^2+y^2+z}$

$f: A \longrightarrow \mathbb{R}$

$A = \mathbb{R}^3 - \{(0,0,0)\}$ Range is $\mathbb{R} - \{0\}$

$f: (x,y,z) \longmapsto \dfrac{1}{x^2+y^2+z}$

f is a scalar-valued function since $m=1$.

E.g. 2] Consider $g(\vec{x}) = g(x_1, x_2, x_3, x_4) = \langle x_1 x_4, \sqrt{x_1^2+x_2^2+x_3^2} \rangle$

$g: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$.

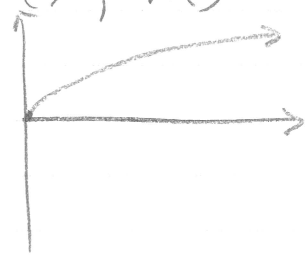
g is a vector-valued function, since $m=2>1$.

How can we describe these?

With graphs!

Take $f(x) = \sqrt{x}$ $f: U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

the graph is all the points $(x, \sqrt{x}) \in \mathbb{R}^n$



We generalize this idea to higher dimensions...

Graphs

Defn: Graph of a function
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Define the graph of of f to be the subset of $\mathbb{R}^{n+1}$ consisting of all the points $(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots x_n))$ in $\mathbb{R}^{n+1}$ for $(x_1, x_2, \dots x_n)$ in U. In symbols,

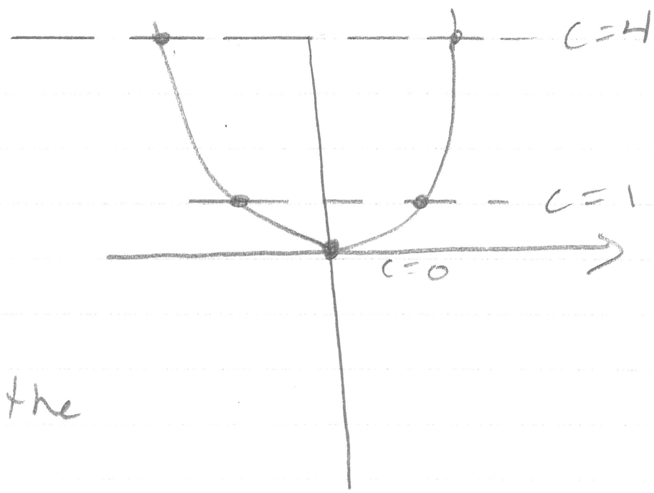
$$\text{graph}(f) = \{(x_1, x_2, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n) \in U\}$$

If $n=1$, these are the 2-D ($\mathbb{R}^2$) graphs we are used to.
If $n=2$, these are surfaces in $\mathbb{R}^3$.

Any higher is hard to visualize...

Ex1) Consider $f(x) = x^2$.
We can set $f(x) = 0$ to get $x = 0$, the x-intercept.
We can set $f(x) = 1 = x^2 \Rightarrow x = \pm 1$ to get where $f(x) = 1$.
Likewise, $f(x) = 4$, $x = \pm 2$.

$f(x) = -10$?
↳ doesn't happen.



"What level is the function?"

Level Sets

Defn: Level Sets
Let $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and let $c \in \mathbb{R}$.
Then the level set of value c is defined to be the set of those points $x = (x_1, x_2, \dots, x_n) \in U$ at which $f(x_1, x_2, \dots, x_n) = c$ or $f(x) = c$.
If $n=2$, we say level curve.
If $n=3$, we say level surface.
In symbols, the level set of value c is

Like a contour map!

$$L_c = \{ x \in U \mid f(x) = c \} \subset \mathbb{R}^n.$$

Level set is always in the domain space.

Ex2 Consider $f(x,y,z) = x^2 + y^2 + z^2$

$f(x,y,z) = 1$ is a sphere of radius 1.
$x^2 + y^2 + z^2 = 1$

$f(x,y,z) = 4$, is a sphere of radius 4.

$f(x,y,z) = 0$, is just the point $(0,0,0)$.

$f(x,y,z) = -$ number, is empty set.

Ex3 $f(x,y) = 7$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = c$, when $c = 7$, it's the xy-plane $z = 7$.
When $c \neq 7$, it's empty.

Ex 4) Consider $f(x,y) = x + y + 4$, think $(z = x + y + 4)$

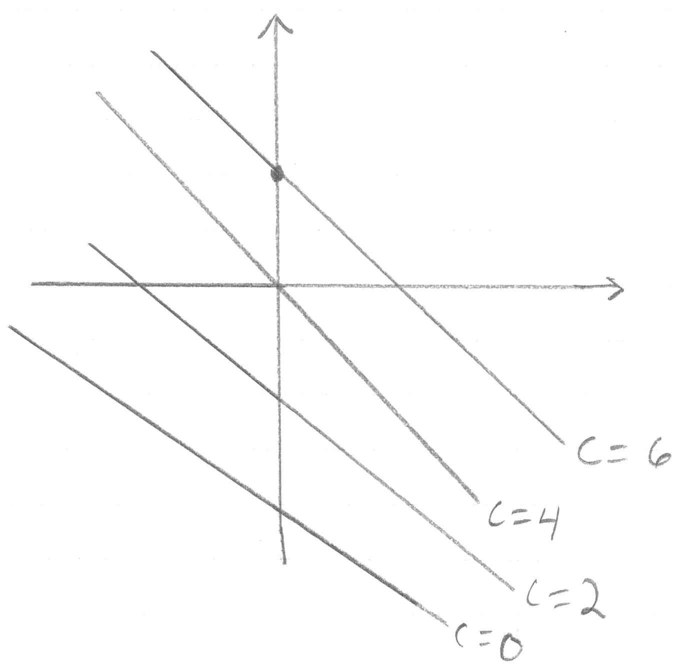
This is a plane, $\vec{n} = \langle 1, 1, -1 \rangle$.

Level curve $0 = x + y + 4$ $\quad y = -x - 4$
how the function intersects the xy-plane.

z-intercept, when $x = y = 0$ is $(0, 0, 4)$

In general, $x + y + 4 = c$

$$L_c = \{ (x,y) \in \mathbb{R}^2 \mid y = -x + (c-4) \} \subset \mathbb{R}^2$$



(← all those are parallel techically.)

Ex 5] Describe $f(x,y) = x^2+y^2$ $f: \mathbb{R}^2 \to \mathbb{R}$

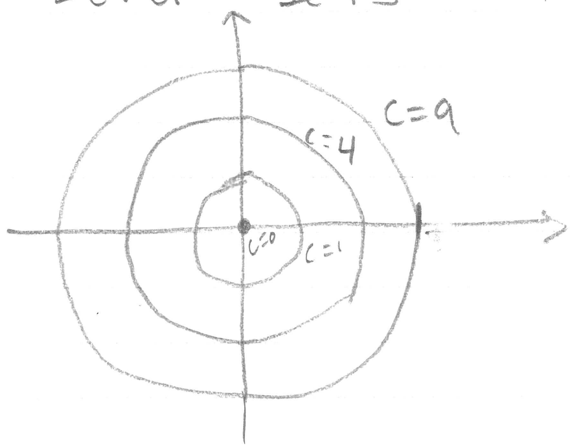
$L_c = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = c \}$

$c<0$ is empty set.

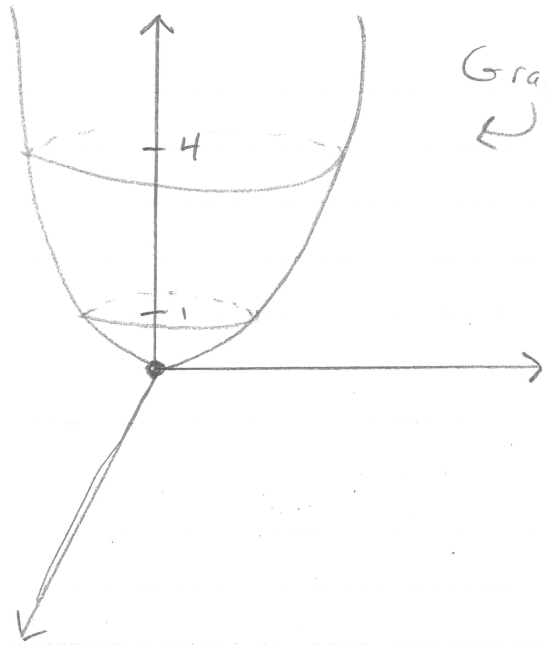
$c=0$ is the point $(0,0)$

$c>0$ are circles of radius $\sqrt{c}$

Level Sets



Graph(f)



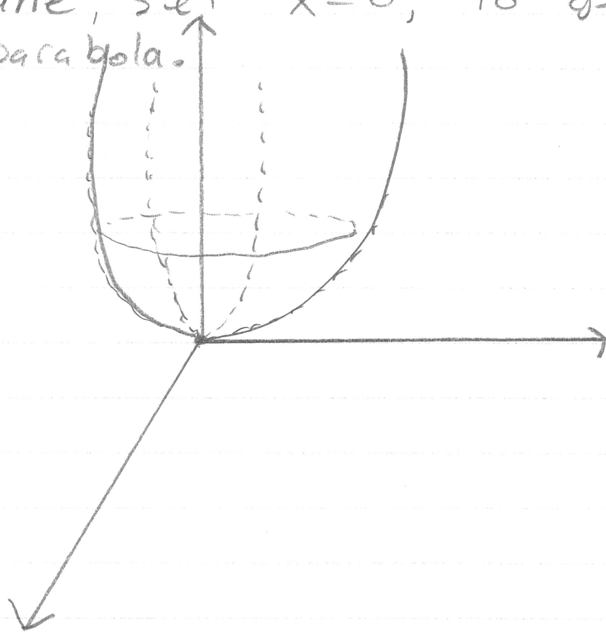
This is called a <u>paraboloid</u>.

Sections

(Another analogue of intercepts)

__Defn:__ Section
A section of the graph of f is the intersection of the graph and a vertical plane. Either the $\underset{P_1}{\underline{\underline{xz}}}$-plane or $\underset{P_2}{\underline{\underline{yz}}}$-plane.

__Ex 5__ cont.
What do the sections of $f(x,y) = x^2 + y^2$ look?
For the xz-plane, set $y=0$, to get
$z = x^2 \rightarrow$ parabola
For the yz-plane, set $x=0$, to get
$z = y^2 \rightarrow$ parabola.



Ex 6) Describe and sketch the graph of $f(x,y) = x^2 - y^2$.

$L_c = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = c\}$.

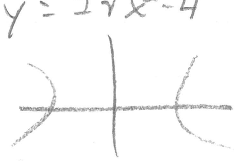
When $c = 0$ $x^2 - y^2 = 0 \Rightarrow y = \pm x$

When $c = 1$, $x^2 - y^2 = 1 \Rightarrow y = \pm\sqrt{x^2 - 1}$

↳ a Hyperbola. openning left/right

When $c = 4$, $x^2 - y^2 = 4 \Rightarrow y = \pm\sqrt{x^2 - 4}$

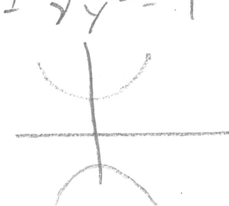
a Hyperbola



when $c = -1$, $x^2 - y^2 = -1$, $x = \pm\sqrt{y^2 - 1}$

Hyperbola

openning downward.



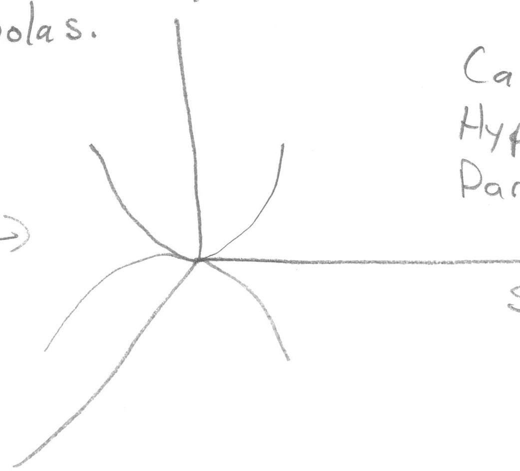
What about sections?

xz-plane: $f(x,0) = x^2$ parabola

yz-plane: $f(0,y) = -y^2$ parabola

Our level sets are Hyperbolas and Sections are parabolas.

I lack the artistic ability.



Called Hyperbolic Paraboloid or Saddle.

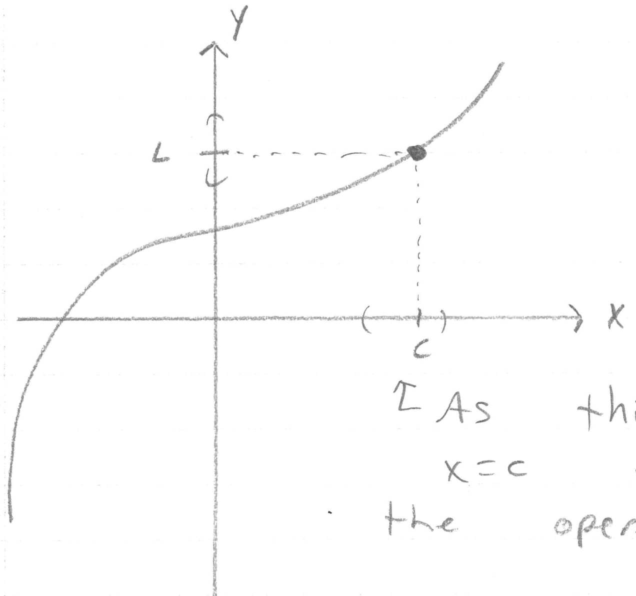
Think pringle's chip.

or saddle of horse.

§2.2 Limits & Continuity

Recall our single variable calculus limit.

$\lim_{x \to c} f(x) = L$ if and only if $|x - c| \to 0$ then $|f(x) - L| \to 0$.



As this open interval around $x=c$ gets smaller, so does the open interval around $y=L$

Left-Hand Limits vs Right-Hand Limits

$\lim_{x \to c^-} f(x)$ $\lim_{x \to c^+} f(x)$

Recall: $\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$

Left and right limit must agree for the overall limit.

Continuity in single variable:

$$\lim_{x \to c} f(x) = f(c)$$

The limit and function value both exist and agree.

Properties of Limits (Limit Laws)

Suppose $\lim_{x \to c} f(x) = L$ & $\lim_{x \to c} g(x) = K$ for functions f, g and constants γ, c, L, K.

Then:

1. $\lim_{x \to c} (f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = L \pm K$

2. $\lim_{x \to c} \gamma f(x) = \gamma \lim_{x \to c} f(x) = \gamma L$

3. $\lim_{x \to c} (f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = K \cdot L$

4. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{L}{K}$ for $K \neq 0$.

Examples of continuous functions:

- Polynomials $x^2 - 8$
- Rational functions $\frac{1}{x}$ (on it's domain)
- Trig $\sin(x)$
- Exponential e^x
- Logs $\ln(x)$ (on it's domain)

Limits

$\lim_{x \to c} f(x) = L \iff$ if $|x-c| \to 0$ then $|f(x) - L| \to 0$

absolute value measures distance in $\mathbb{R}$.

What measures distance in $\mathbb{R}^n$?

$$\|\vec{b} - \vec{a}\| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + \cdots (b_n-a_n)^2}$$

<u>Defn:</u> Limit
<u>Let</u> $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. We say a vector $\vec{b} \in \mathbb{R}^m$ is the limit as $\vec{x}$ approaches $\vec{c} \in U$, when $\vec{x} \neq \vec{c}$

If $\|\vec{x} - \vec{c}\| \to 0$ then $\|f(\vec{x}) - \vec{b}\| \to 0$.

In which case we write $\lim_{\vec{x} \to \vec{c}} f(\vec{x}) = \vec{b}$.

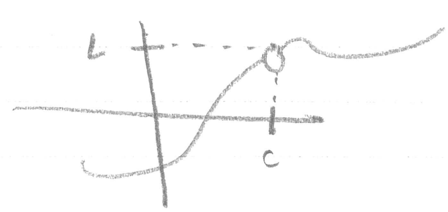
If else, we say the limit doesn't exist.

<u>Special Cases:</u> $n = 2$ or 3 & $m = 1$.

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

$[\ (x,y,z) \to (a,b,c) \quad f(x,y,z)\ ]$ $n = 3$

Remember the function doesn't have to be defined at $x = c$ for the limit to exist.



Thm: If $f(x,y) \to L_1$ along a path C_1
and $f(x,y) \to L_2$ along a path C_2
as $(x,y) \to (a,b)$, where $L_1 \neq L_2$,

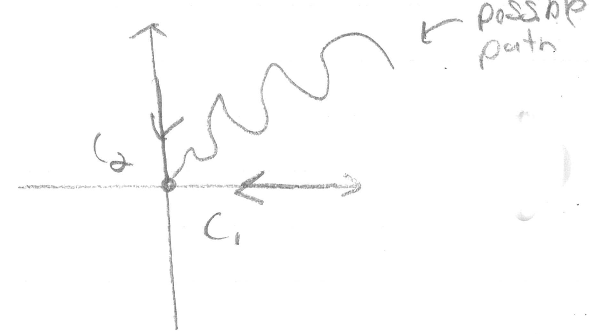
then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = \text{DNE}$.

Thm: Limit Laws hold true.

Ex 1) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{DNE}$.

Let's pick the first path along the x-axis.
So $y=0$ & $x \to 0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$



For the 2nd path let's approach along the y-axis.
So $x=0$ & $y \to 0$

$$\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$$

"pinched along x-axis & y-axis"

Limits

Ex 2) Does $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

Let first approach along the x-axis, so $y=0$.

$\lim_{(x,0)} \frac{0}{x^2} = 0.$

Next, try along y-axis, so $x=0$

$\lim_{(0,y)\to(0,0)} \frac{0}{x^2+y^2} = 0.$

So the limit exists? NO,
Has to agree for all paths.
Let try along $y=x$

$\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{1}{2} = \frac{1}{2}.$

Thus the limit doesn't exist.

"Pinched along $y=x$"

Ex 3 | Does $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

Checking one path at a time seems a bit ineffcient, let's try a bunch at the same time...

Let's approach on all lines $y = mx$, different m different path of approach.

(If $m=0$, this the x-axis path)

$$\lim_{(x,mx)\to(0,0)} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{(x,mx)\to(0,0)} \frac{m^2x^3}{x^2+m^4x^4} = 0$$

But sadly, approach on parabola $x = y^2$.

$$\lim_{(y^2,y)} \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

Thus $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \text{DNE}$.

"parabolically pinched"

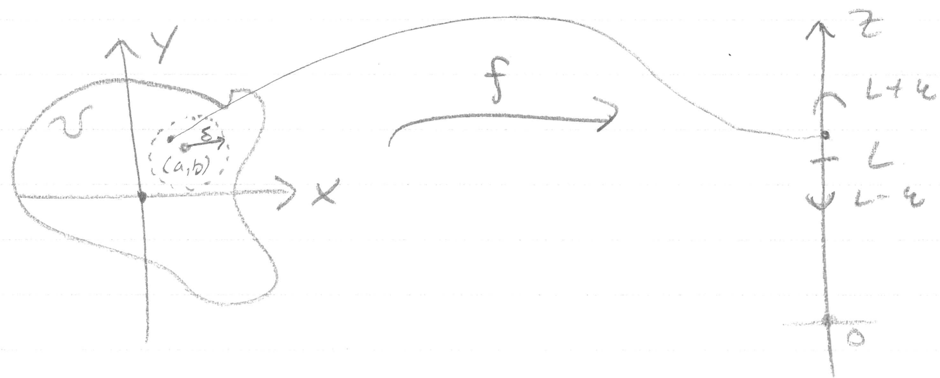
Limits that do exist

<u>**Defn:**</u> $\varepsilon-\delta$ Definition (for $n=2$ & $m=1$)

Let $f(x,y)$ be a function $f: U \subset \mathbb{R}^2 \to \mathbb{R}$.

Then we say $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$

if for all $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that if $(x,y) \in U$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.



This definition is rather hard to work with.

To compute our limits easier, we need continuity or to convert back to single variable calculus.

<u>Ex 4</u> Does $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ exist?

Recall polar coordinates $x^2+y^2 = r^2 := t$
If $(x,y) \to (0,0)$ then $r \to 0$.
If $r \to 0$, then $r^2 \to 0$.

So this is really

$$\lim_{t\to 0} \frac{\sin(t)}{t} = \frac{0}{0} \quad \ddot{\frown}$$

$\Rightarrow$ L'Hopital's $\lim_{t\to 0} \frac{\cos(t)}{1} = 1$

So, $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$.

<u>Rmk</u>: We used a substuition $x^2+y^2 = t$.

<u>Ex 5</u> $\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$

$y \neq 0$, or else not in domain!

$$\lim_{(x,y)\to(0,0)} x \cdot \left(\frac{e^{xy}-1}{xy}\right) = \Big(\underbrace{\lim_{(x,y)\to(0,0)} x}_{=0}\Big) \cdot \left(\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}\right)$$

set $xy = t$

$$\lim_{t\to 0} \frac{e^t - 1}{t} = \lim_{t\to 0} \frac{e^t}{1} = 1$$

So $\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = 0 \cdot 1 = 0$.

$$\begin{array}{ccc} \vec{x}_0 & & \vec{b} \\ \Downarrow & & \Downarrow \\ f : \mathbb{R}^n & \to & \mathbb{R}^m \end{array}$$

Properties of Limits

Thm 2 1. If $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}_1$ & $\lim_{\vec{x} \to x_0} f(\vec{x}) = \vec{b}_2$

then $\vec{b}_1 = \vec{b}_2$. Limits are unique.

thm 3

2. If $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}$, then $\lim_{\vec{x} \to \vec{x}_0} cf(\vec{x}) = c\vec{b}$.

3. If $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}_1$ & $\lim_{\vec{x} \to \vec{x}_0} g(\vec{x}) = \vec{b}_2$, then

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) \pm \lim_{\vec{x} \to \vec{x}_0} g(\vec{x})$$

4. If $m = 1$, $\lim_{\vec{x} \to x_0} f(\vec{x}) = b_1$, $\lim_{\vec{x} \to x_0} g(\vec{x}) = b_2$

then $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) g(\vec{x}) = b_1 \cdot b_2$.

5. If $m = 1$, $\lim_{\vec{x} \to x_0} f(\vec{x}) = b \neq 0$, & $f(\vec{x}) \neq 0$

for all $\vec{x}$ in domain, then $\lim_{\vec{x} \to x_0} \frac{1}{f(\vec{x})} = \frac{1}{b}$.

Vector-Valued Function:

New ↳ 6. If $F(\vec{x}) = \langle f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x}) \rangle$

where each $f_i : A \to \mathbb{R}$, for $i = 1, 2, \dots, m$

then $\lim_{\vec{x} \to \vec{x}_0} F(\vec{x}) = \vec{b} = \langle b_1, b_2, \dots, b_m \rangle$

if and only if $\lim_{\vec{x} \to \vec{x}_0} f_i(\vec{x}) = b_i$ for $i = 1, 2, \dots, m$.

Continuity

no holes or breaks

Defn: Continuity
Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function with domain A. Let $\vec{x}_0 \in A$. We say f is continuous at $\vec{x}_0$ if and only if
$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0).$$

- If f is continuous at all $\vec{x} \in A$, then f is continuous on A.
- If f is not continuous, we say discontinuous.

Ex 6 Evaluate $\displaystyle\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

This is a polynomial in 2 variables, hence continuous, so

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$$
where $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$

$$f(1,2) = 1^2 \cdot 2^3 - 1^3 2^2 + 3 \cdot 1 + 2 \cdot 2$$
$$= 8 - 4 + 3 + 4 = 11$$

So $\displaystyle\lim_{(x,y)\to(1,2)} f(x,y) = 11$

Examples of continuous functions

1. Polynomials in n-variables
   ↳ everywhere continuous. $(\mathbb{R}^n)$

2. Rational Functions in n-variables
   ↳ everywhere in domain.

e.g. $f(x,y) = \dfrac{xy}{x^2+y^2}$ $(0,0)$ not in domain.
$\mathbb{R}^2 - \{(0,0)\}$

Thm 5: Continuity of Compositions

Let $g : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $f : B \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose $g(A) \subset B$, so that $f \circ g$ is defined on A. If g is continuous at $\vec{x}_0 \in A$ and f is continuous at $\vec{y}_0 = g(\vec{x}_0)$, then $f \circ g$ is continuous at $\vec{x}_0$.

Rmk: $(f \circ g)(\vec{x}) = f(g(\vec{x}))$

Ex 71 Is $f(x,y) = \underbrace{(x^2+y^2+z^2)^{42}} + \underbrace{\sin(z^9)}$ continuous?

t^{42}

polynomial $\xrightarrow{\text{cont}}$ $w = z^9$

polynomial → cont.

trig (w) → cont

cont + cont. = cont.

So, yes f is continuous.

§2.3 Partial Derivatives

Recall Derivative in single variable:

$$\frac{dy}{dx} = \frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Defn: Partial Derivative

Let $U \subset \mathbb{R}^n$ be an open set. Suppose $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is a real-valued function. Then $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n}$ the partial derivatives of f with respect to the 1st, 2nd, ..., nth variable, are the real-valued functions of n variables, which, at the point $X = (x_1, x_2, \dots x_n)$ are defined by

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \dots x_n) = \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_j + h, \dots x_n) - f(x_1, \dots, x_n)}{h}$$

($\vec{x}$ as a vector) $$= \lim_{h \to 0} \frac{f(\vec{x} + h e_j) - f(\vec{x})}{h}$$

if the limits exist, where $1 \le j \le n$. The domain of $\frac{\partial f}{\partial x_j}$ is the set where the limit exists.

$$\boxed{\text{In other words, } \frac{\partial f}{\partial x_j} \text{ is just the derivative of } f \text{ with respect to the variable } x_j, \text{ with the other variables held constant.}}$$

Special Cases:

- $f(x,y) \longrightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \quad (f_x, f_y)$

- $f(x,y,z) \longrightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

- If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$

$$f(x_1, x_2, \dots, x_n) = \left(f_1(x_1, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots f_m(x_1, \dots x_n) \right)$$

We can have $\frac{\partial f_m}{\partial x_n}$

partial of the m-th component with respect to the n-th variable.

<u>Ex 1</u>) If $f(x,y) = xy^3 + x^2 + 8$

For $\frac{\partial f}{\partial x}$ hold y constant.

$$\boxed{\frac{\partial f}{\partial x} = y^3 + 2x}$$

For $\frac{\partial f}{\partial y}$, hold x constant

$$\boxed{\frac{\partial f}{\partial y} = 3xy^2}$$

$$\underbrace{}_{}\;\; f(x,y)$$

Ex 2 If $z = \overbrace{y\sin(x) + \sin(xy) + \frac{y}{x}}^{f(x,y)}$

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

Fix y, y is constant
chain rule

$$\frac{\partial z}{\partial x} = y\cos(x) + y\cos(xy) - \frac{y}{x^2}$$

Hold x constant.

$$\frac{\partial z}{\partial y} = \sin(x) + x\cos(xy) + \frac{1}{x}$$

Ex 3 If $f(x,y) = \frac{4xy}{\sqrt{x^2+y^2}}$, find $\frac{\partial f}{\partial y}(3,4)$

Quotient rule

$u = 4xy$ $\quad u_y = 4x$

$v = (x^2+y^2)^{\frac{1}{2}}$ $\quad v_y = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$

$$= y(x^2+y^2)^{-\frac{1}{2}}$$

$$\frac{\partial f}{\partial y} = \frac{4x(x^2+y^2)^{\frac{1}{2}} - 4xy^2(x^2+y^2)^{-\frac{1}{2}}}{(x^2+y^2)}$$

$$\frac{(x^2+y^2)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}}}$$

$$\frac{\partial f}{\partial y} = \frac{4x(x^2+y^2) - 4xy^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial f}{\partial y} = \frac{4x^3 + \cancel{4xy^2} - \cancel{4xy^2}}{(x^2+y^2)^{3/2}} = \frac{4x^3}{(\sqrt{x^2+y^2})^3}$$

$$\frac{\partial f}{\partial y}(3,4) = \frac{4 \cdot 3^3}{(\sqrt{3^2+4^2})^3} = \frac{4 \cdot 27}{125} = \frac{108}{125}$$

What goes wrong....

Ex 4 Let $f(x,y) = \sqrt{x}\sqrt{y}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}}$$

$$\frac{\partial f}{\partial x}(0,0) = \text{DNE?} \qquad \frac{\partial f}{\partial y}(0,0) = \text{DNE?}$$

Go to the definition:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Thus by definition $\frac{\partial f}{\partial x}(0,0) = 0$

(Ditto with $\frac{\partial f}{\partial y}$)

$f(x,y)$ has a "crinkle" near $(0,0)$
"cusp"

Despite $\left.\frac{\partial f}{\partial x}\right|_{(0,0)}$, $\left.\frac{\partial f}{\partial y}\right|_{(0,0)}$ existing...

$f(x,y)$ is not differentiable there.

[Recall in single variable, f was differentiable at a if $f'(a)$ exists]

One goal in calculus was to find the tangent line $y = f'(a)(x-a) + f(a)$.

We want to find the tangent plane, something like this: $z = ax + by + c$

$$\frac{\partial z}{\partial x} = a \quad \& \quad \frac{\partial z}{\partial y} = b$$

Thus, the tangent plane of $f(x,y)$ @ (x_0, y_0) is

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

This is linear approximation. [at $(x_0, y_0, f(x_0, y_0))$]

In order for a function to be differentiable we'll require this linear approximation to be "good."

Let's examine this in single-variable:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

Sub $x = a + h$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) = \lim_{x \to a} \underbrace{f'(a)}_{\text{constant}}$$

So

$$\lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} - f'(a)\right] = 0$$

$$\lim_{x \to a} \left[\frac{f(x) - f(a) - f'(a)(x-a)}{x - a}\right] = 0$$

$$\lim_{x \to a} \left[\frac{f(x) - \overbrace{(f'(a)(x-a) + f(a))}^{\text{tangent line}}}{x - a}\right] = 0$$

Defn: Differentiable $(n=2,\ m=1)$

Let $f: \mathbb{R}^2 \to \mathbb{R}$. We say f is differentiable at (x_0, y_0), if $\underbrace{\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ exist at } (x_0, y_0)}_{①}$

AND if

$$0 = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\|(x,y)-(x_0,y_0)\|} = 0$$

Or, the tangent plane is a "good" approximation.

Ex 5 What is the tangent plane to the graph of $f(x,y) = x^3 + y^4 + 2e^{xy}$ at $(1,0)$?

$f(1,0) = 1 + 0 + 2 = 3$

$\frac{\partial f}{\partial x} = 3x^2 + 2ye^{xy} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,0)} = 3$

$\frac{\partial f}{\partial y} = 4y^3 + 2xe^{xy} \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,0)} = 2$

So, my tangent plane is

$$z = 3 + 3(x-1) + 2(y-0) = 3 + 3x - 3 + 2y$$
$$= 3x + 2y$$

(row vector) **Notation:** Let's write $Df(x_0,y_0)$ or $\nabla f(x_0,y_0)$ (nabla, del) as the row matrix $\left[\frac{\partial f}{\partial x}(x_0,y_0) \quad \frac{\partial f}{\partial y}(x_0,y_0)\right]$

So, our tangent plane is:

$$z = f(x_0,y_0) + Df(x_0,y_0)\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

matrix multiplication!

$$= f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$$

Defn: Differentiable (n-variables, m Functions)
(Let U be an open set in $\mathbb{R}^n$). Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function. We say that f is differentiable at $\vec{x_0} \in U$ if the partial derivatives of f exist at $\vec{x_0}$ and if

$$\lim_{\vec{x} \to \vec{x_0}} \frac{\| f(\vec{x}) - f(\vec{x_0}) - Df(\vec{x_0})(x - x_0) \|}{\| \vec{x} - \vec{x_0} \|} = 0$$

$Df(\vec{x_0}) \left(= T = J(\vec{x_0}) \right)$ is the $m \times n$ matrix with matrix elements $\frac{\partial f_i}{\partial x_j}$ evaluated at $\vec{x_0}$ and $Df(\vec{x_0}) \cdot (\vec{x} - \vec{x_0})$ is matrix multiplication. We say $Df(\vec{x_0})$ the derivative of f at $\vec{x_0}$.

Special Case:

$m=1$ $\quad Df(\vec{x_0}) = \nabla f(\vec{x_0})$

$$= \left[\frac{\partial f}{\partial x_1}(\vec{x_0}) \quad \frac{\partial f}{\partial x_2}(\vec{x_0}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\vec{x_0}) \right]$$

When we think of this $1 \times n$ matrix as a vector, we call it the <u>gradient</u> of f or grad(f).

Worst Case: (n, m)

$$Df(\vec{x_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Matrix of partial Derivatives.

Ex 6 Compute $Df(x,y)$ for

$$f(x,y) = (\underbrace{ye^x}_{f_1},\ \underbrace{x^2 + \cos(2y)}_{f_2})$$

$$Df(x,y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\[2ex] \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} ye^x & e^x \\[2ex] 2x & -2\sin(2y) \end{bmatrix}$$

Ex 7 Compute $Df(x,y,z)$ for

$$f(x,y,z) = (\underbrace{-ye^z}_{f_1},\ \underbrace{xe^z}_{f_2})$$

$$Df(x,y,z) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\[2ex] \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -e^z & -ye^z \\[2ex] e^z & 0 & xe^z \end{bmatrix}$$

Gradient & Thms

<u>Ex 8]</u> Let $f(x, y, z) = xyz + xe^y$
Find grad (f).

$$\text{grad}(f) = \left\langle yz + e^y,\ xz + xe^y,\ xy \right\rangle$$

Two important Theorems:

<u>Thm 8:</u>
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{x_0} \in U$, then f is continuous at $\vec{x_0}$.

"Smooth enough" for a tangent plane, must mean "smooth" = (continuous)

To actually check differentiability, the definition is rather hard...

<u>Thm 9:</u>
Let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Suppose the partial derivatives $\frac{\partial f_i}{\partial x_j}$ of f all exist

AND are continuous "near" $\vec{x} \in U$, then f is differentiable at $\vec{x}$.

$$\text{Cont partials} \xRightarrow{\text{thm a}} \text{Differentiable} \xRightarrow{\text{Defn of partial}} \text{Partials exist}$$

§2.4 Intro to Paths & Curves

Often when we say "curve" we mean something we can draw on the xy-plane.
Let's think of this instead as a <u>path</u> starting somewhere and ending somewhere.

<u>Ex 11</u> Consider the point (x_0, y_0, z_0) in the direction of $\vec{v}$

$$\vec{c}(t) = (x_0, y_0, z_0) + t\vec{v}$$

for $t \in \mathbb{R}$.
This is a path.
If we restrict $t \in [a, b]$, we start at $\vec{c}(a)$ and end at $\vec{c}(b)$

<u>Defn:</u> Paths & Curves
A path in $\mathbb{R}^n$ is a map $\vec{c} : [a, b] \to \mathbb{R}^n$. The collection C of points $\vec{c}(t)$ as t varies in $[a, b]$ is called a curve, and $\vec{c}(a)$ & $\vec{c}(b)$ are it's endpoints. The path $\vec{c}$ parameterizes the curve C.

<u>Special Case:</u>

- Parametric from calc 1
  (path in plane)
- If $\vec{c}$ is a path in $\mathbb{R}^3$, or in space,

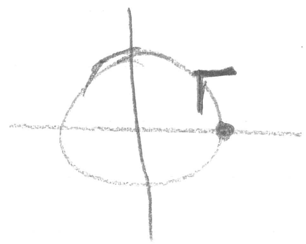
$$\vec{c}(t) = (x(t), y(t), z(t))$$

Component functions.

Ex 2] Take the unit circle $x^2+y^2=1$

$\vec{c} : \mathbb{R} \to \mathbb{R}^2$

$$\vec{c}(t) = (\cos(t), \sin(t)) \qquad 0 \le t \le 2\pi$$



$$\vec{c}(0) = \vec{c}(2\pi) = (0,0)$$

$$\vec{c}(\tfrac{\pi}{2}) = (0,1)$$

$$\vec{c}(\pi) = (-1,0)$$

Also the same curve we can parameterize as follows:

$$\vec{c}(t) = (\cos(2t), \sin(2t)) \qquad 0 \le t \le \pi$$

Rmk: Different paths may parameterize the same curve.

Ex 3] $\vec{c}(t) = (t, t^2)$ traces out a parabola (if $t \in \mathbb{R}$)

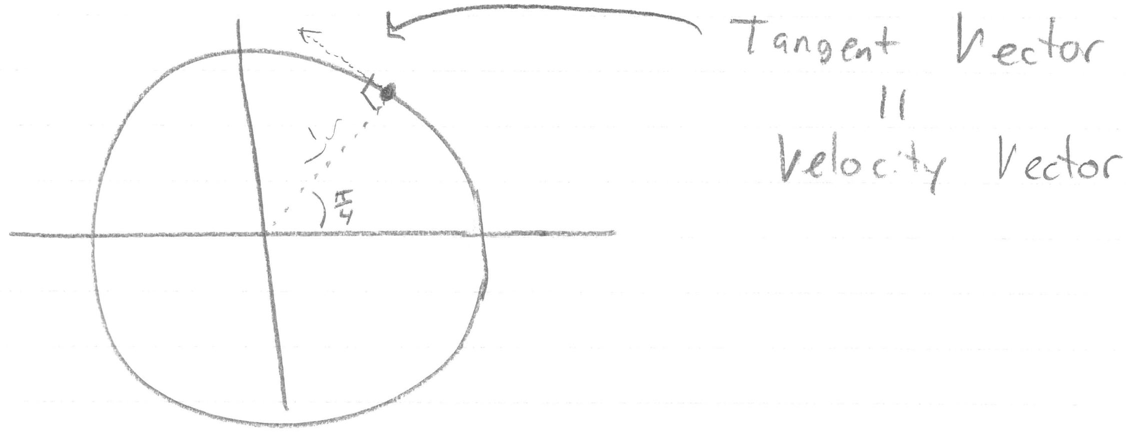
$$\begin{cases} x = t \\ y = t^2 \end{cases} \implies y = x^2$$

$$\vec{c}(-1) = (-1, 1)$$
$$\vec{c}(0) = (0, 0)$$
$$\vec{c}(1) = (1, 1)$$

Velocity & Tangent Vectors

Think about the unit circle again

$\vec{c}(t) = (\cos(t), \sin(t)) \qquad 0 \le t \le 2\pi$



<u>Defn:</u> Velocity Vector

If $\vec{c}$ is a path and it is differentiable, we say $\vec{c}$ is a differentiable path. The velocity of $\vec{c}$ at time t is

$$\vec{c}\,'(t) = \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}$$

The speed would be $\|c'(t)\|$.
The velocity $\vec{c}\,'(t)$ is a vector tangent to the path $\vec{c}(t)$ at time t. If C is a curve traced out by $\vec{c}$ and if $\vec{c}\,'(t) \neq \vec{0}$, then $\vec{c}\,'(t)$ is a vector tangent to the curve at the point $\vec{c}(t)$.

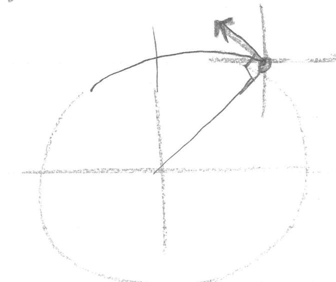
Special cases:

- If $\vec{c}(t) = (x(t), y(t)) = x(t)\,\hat{i} + y(t)\,\hat{j}$
  then $\vec{c}\,'(t) = (x'(t), y'(t)) = x'(t)\,\hat{i} + y'(t)\,\hat{j}$

- If $\vec{c}(t) = (x(t), y(t), z(t)) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$
  then $\vec{c}\,'(t) = (x'(t), y'(t), z'(t)) = x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}$.

Ex 4 Compute the tangent vector to the path $\vec{c}(t) = (\cos(t), \sin(t))$ at $t = \pi/4$.

$$\vec{c}\,'(t) = (-\sin(t), \cos(t))$$
$$\vec{c}\,'(\tfrac{\pi}{4}) = \left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right)$$



Ex 5 Compute the tangent vector to the path $\vec{c}(t) = (t^2, \frac{1}{t}, e^t)$ at $t=2$.

$$\vec{c}\,'(t) = (2t, -\tfrac{1}{t^2}, e^t)$$

$$c'(2) = (4, -\tfrac{1}{4}, e^2)$$

We can take this tangent vector and get a line.

Defn: Tangent line to a path
If $\vec{c}(t)$ is a path, and if $\vec{c}\,'(t_0) \neq \vec{0}$, the equation of its tangent line at the point $\vec{c}(t_0)$ is

$$\vec{l}(t) = \vec{c}(t_0) + (t - t_0)\,\vec{c}\,'(t_0)$$

(Note $\vec{l}(t_0) = \vec{c}(t_0)$)

Ex 6 | Consider the path

$$\vec{c}(t) = (1+t^3)\,\hat{i} + te^{-t}\,\hat{j} + \sin(2t)\,\hat{k}$$

Find the tangent line at $t=0$.

$$\vec{c}\,'(t) = (3t^2,\ e^{-t} - te^{-t},\ 2\cos(2t))$$

$$\vec{c}\,'(0) = (0,\ 1,\ 2)$$

$$\vec{c}(0) = (1,\ 0,\ 0)$$

$$\vec{\ell}(t) = (1,0,0) + (t-0)(0,1,2)$$

$$= (1,0,0) + t(0,1,2)$$

or

$$\begin{cases} x = 1 \\ y = t \\ z = 2t \end{cases}$$

§2.5 Properties of the Derivative

In higher dimensions, most of our properties work how we'd expect. The chain rule is one of the exceptions, it's a bit harder.

<u>Thm 10:</u> Sums / Differences, Products, Quotient Rule

i) Constant Multiple Rule - Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{x}_0$ and $c \in \mathbb{R}$. Then $h(\vec{x}) = c f(\vec{x})$ is differentiable at $\vec{x}_0$ and

$$Dh(\vec{x}_0) = c\, Df(\vec{x}_0).$$ [Equality of Matrices]

ii) Sum/Difference Rule - Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{x}_0$. Then $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$ is differentiable at $\vec{x}_0$ and

$$Dh(\vec{x}_0) = Df(\vec{x}_0) + Dg(\vec{x}_0)$$

iii) Product Rule / Quotient Rule

Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ $(m = 1)$ be differentiable at $\vec{x}_0$ and let $p(\vec{x}) = f(\vec{x})\, g(\vec{x})$ and $q(\vec{x}) = \dfrac{f(\vec{x})}{g(\vec{x})}$ (when $g(\vec{x}) \neq 0$)

Then $p(\vec{x})$ & $q(\vec{x})$ are differentiable at $\vec{x}_0$ and

$$Dp(\vec{x}_0) = \left(Df(\vec{x}_0)\right) g(\vec{x}_0) + f(\vec{x}_0)\left(Dg(\vec{x}_0)\right)$$

$$Dq(\vec{x}_0) = \frac{Df(\vec{x}_0)\, g(\vec{x}_0) - f(\vec{x}_0)\left(Dg(\vec{x}_0)\right)}{[g(\vec{x}_0)]^2}$$

Ex 11 Verify the quotient rule for $f(x,y,z) = x^2+y^2+z^2$ & $g(x,y,z) = x^2+1$.

So $\quad q(x) = \dfrac{x^2+y^2+z^2}{x^2+1} = \dfrac{x^2}{x^2+1} + \dfrac{y^2}{x^2+1} + \dfrac{z^2}{x^2+1}$

Directly:

$$Dq(x) = \left[\frac{2x(x^2+1) - 2x(x^2+y^2+z^2)}{(x^2+1)^2}, \frac{2y}{x^2+1}, \frac{2z}{x^2+1}\right]$$

Or $\quad Df = [2x, 2y, 2z] \qquad Dg = [2x, 0, 0]$

So

$$Dq(x) = \frac{(Df)g - f(Dg)}{g^2} = \frac{[2x, 2y, 2z](x^2+1) - (x^2+y^2+z^2)[2x,0,0]}{(x^2+1)^2}$$

$$= \frac{1}{(x^2+1)^2}\Big[[2x(x^2+1), 2y(x^2+1), 2z(x^2+1)] - [2x(x^2+y^2+z^2), 0, 0]\Big]$$

$$= \frac{1}{(x^2+1)^2}\left[2x(x^2+1) - 2x(x^2+y^2+z^2), 2y(x^2+1), 2z(x^2+1)\right]$$

$$= \left[\frac{2x(x^2+1) - 2x(x^2+y^2+z^2)}{(x^2+1)^2}, \frac{2y}{x^2+1}, \frac{2z}{x^2+1}\right]$$

✓

Chain Rule

Recall Chain Rule for single-variable

If $F(x) = f(g(x)) = \sin(x^3)$

then
$$F'(x) = f'(g(x)) \cdot g'(x) = \cos(x^3) \cdot 3x^2$$

Or say, $z = f(y)$ & y is a function of x

$y = g(x)$

$z = f(g(x))$

then
$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(g(x))\, g'(x).$$

Thm 11: Chain Rule

(Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets.)

Let $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ & $f: V \subset \mathbb{R}^m \to \mathbb{R}^p$ be given functions such that g maps U into V, so that $f \circ g$ is defined. Suppose g is differentiable at $\vec{x_0}$ and f is differentiable at $\vec{y_0} = g(\vec{x_0})$. Then $f \circ g$ is differentiable at $\vec{x_0}$ and

$$\boxed{D(f \circ g)(\vec{x_0}) = Df(g(\vec{x_0}))\, Dg(\vec{x_0})}$$

Matrix Multiplication.

Special Case 1:

Suppose $\vec{c}: \mathbb{R} \to \mathbb{R}^3$ is a differentiable path and $f: \mathbb{R}^3 \to \mathbb{R}$. Let $h(t) = f(\vec{c}(t)) = f(x(t), y(t), z(t))$, where $\vec{c}(t) = (x(t), y(t), z(t))$. Then

$$Dh(t) = \frac{dh}{dt} = \nabla f(\vec{c}(t)) \cdot c'(t)$$
$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \cdot \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$$
$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Ex21 Verify Chain Rule for
$f(x,y) = x^2y^3$ & $\vec{c}(t) = (3t^3, e^t)$

Directly $h = (f \circ c)(t) = (3t^3)^2 (e^t)^3$
$= 9t^6 e^{3t}$

So $Dh = 54t^5 e^{3t} + 9t^6 \cdot 3e^{3t}$

Or, $\frac{\partial f}{\partial x} = 2xy^3$ & $\frac{\partial f}{\partial y} = 3x^2y^2$

and $\vec{c}\,'(t) = (9t^2, e^t)$

$\nabla f(\vec{c}) \cdot c' = \left[2(3t^3)(e^{3t}), 3(3t^3)^2 e^{2t}\right] \cdot \left[9t^2, e^t\right]$

$= 54t^5 e^{3t} + 3 \cdot 9t^6 e^{3t}$ ✓

Special Case 2:

Let $f: \mathbb{R}^3 \to \mathbb{R}$ and $g: \mathbb{R}^3 \to \mathbb{R}^3$
$g(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$

Define $h: \mathbb{R}^3 \to \mathbb{R}$ $h(x,y,z) = (f \circ g)(x,y,z)$

$\left[\frac{\partial h}{\partial x}\ \frac{\partial h}{\partial y}\ \frac{\partial h}{\partial z}\right] = Dh = D(f \circ g)(x,y,z) = D(f(u,v,w)) \cdot D(g(x,y,z))$

$$= \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

E.g.
$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

Ex3] Let $f(u, v, w) = u^3 - v^2 + w$

$u(x,y,z) = x^2 y$, $v(x,y,z) = y^3$, $w(x,y,z) = e^{-xz}$

Let $h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$

Find $\frac{\partial h}{\partial x}$ directly & via chain rule.

Directly: $h(x,y,z) = (x^2y)^3 - (y^3)^2 + e^{-xz}$

$$= x^6y^3 - y^6 + e^{-xz}$$

$$\frac{\partial h}{\partial x} = 6x^5y^3 - ze^{-xz}$$

Chain Rule:

$$\frac{\partial f}{\partial u} = 3u^2 \qquad \frac{\partial f}{\partial v} = -2v \qquad \frac{\partial f}{\partial w} = 1$$

$$\frac{\partial u}{\partial x} = 2xy \qquad \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial w}{\partial x} = -ze^{-xz}$$

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= 3(x^2y)^2 \cdot 2xy + -2y^3 \cdot 0 + 1(-ze^{-xz})$$

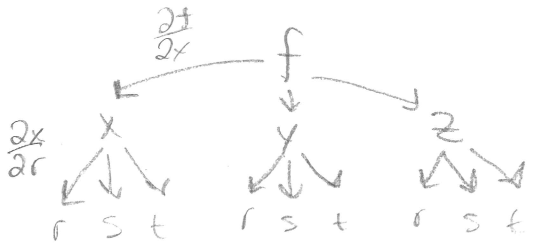
$$= 3x^4y^2 \cdot 2xy - ze^{-xz}$$

$$= 6x^5y^3 - ze^{-xz} \quad \checkmark$$

Ex 4) If $f(x,y,z) = x^4 y + y^2 z^3$

$x(r,s,t) = rse^t$, $y(r,s,t) = rs^2 e^{-t}$, $z(r,s,t) = r^2 s \sin(t)$

Find the value of $\dfrac{\partial f}{\partial s}$ at $(2, 1, 0)$.
(r, s, t)



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial s} = (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t})$$
$$+ (3y^2 z^2)(r^2 \sin(t))$$

$x(2,1,0) = 2$ $\quad$ $y(2,1,0) = 2$ $\quad$ $z(2,1,0) = 0$

$$\frac{\partial f}{\partial s}(2,1,0) = (4\, 2^3 \cdot 2)(2 \cdot e^0) + (2^4 + 2 \cdot 2 \cdot 0^3)(2 \cdot 2 \cdot 1 e^0)$$
$$+ (3\, 2^2\, 0^2)\cancel{(2^2 \sin(0))}$$

$$= (64)(2) + (16)(4) + (0)(0)$$

$$= 192$$

§2.6 Gradients & Directional Derivatives

Recall: Gradient

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable, the gradient of f at (x,y,z) is the vector in space given by $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Or $\nabla f(x,y,z)$, or Df but thought of as a vector.

Ex 1) If $f(x,y,z) = x\sin(yz)$, find $\nabla f(1,3,0)$

$\nabla f = \langle \sin(yz), xz\cos(yz), xy\cos(yz) \rangle$

$\nabla f(1,3,0) = \langle 0, 0, 3 \rangle$

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$

Consider the line $\vec{l}(t) = \vec{x} + t\vec{v}$

Look at $f(\vec{x} + t\vec{v})$, this is f restricted to the line.

How does f change on this line?

How does f change on this line at $\vec{x}$?

Derivative

Defn: Directional Derivative

If $f: \mathbb{R}^3 \to \mathbb{R}$, the directional derivative of f at $\vec{x}$ along the vector $\vec{v}$ is given by

$$\frac{d}{dt} f(\vec{x}+t\vec{v}) \Big|_{t=0} = \lim_{h \to 0} \frac{f(\vec{x}+h\vec{v}) - f(\vec{x})}{h}$$

if this exists.

We (usually) choose $\vec{v}$ to be a unit vector. In this case, we are moving in the direction $\vec{v}$ with unit speed and we say $\frac{d}{dt} f(\vec{x}+t\vec{v})\Big|_{t=0}$ is the directional derivative of f in the direction $\vec{v}$.

Thm 12:

If $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then all directional derivatives exist. The directional derivative at $\vec{x}$ in the direction $\vec{v}$ is given by

$$Df(\vec{x}) \cdot \vec{v} = \text{grad}(f) \cdot \vec{v} = \nabla f(\vec{x}) \cdot \vec{v}$$

$$= \left[\frac{\partial f}{\partial x}(\vec{x})\right] v_1 + \left[\frac{\partial f}{\partial y}(\vec{x})\right] v_2 + \left[\frac{\partial f}{\partial z}(\vec{x})\right] v_3$$

where $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

proof. Let $\vec{c}(t) = \vec{x} + t\vec{v}$, so $f(\vec{x}+t\vec{v}) = f(\vec{c}(t))$

By chain rule, $\frac{d}{dt}\left(f(\vec{c}(t))\right) = \nabla f(\vec{c}(t)) \cdot c'(t)$

$\vec{c}(0) = \vec{x}$ & $\vec{c}\,'(0) = \vec{v}$, so

$$\frac{d}{dt} f(\vec{x}+t\vec{v}) \Big|_{t=0} = \nabla f(\vec{x}) \cdot \vec{v}$$

$\square$

Ex 11 Find the directional derivative of $f(x,y,z) = \sqrt{xyz}$ at $(3,2,6)$ in the direction of $\vec{v} = \langle -1, -2, 2 \rangle$

$$\nabla f = \left\langle \frac{\sqrt{yz}}{2\sqrt{x}}, \frac{\sqrt{xz}}{2\sqrt{y}}, \frac{\sqrt{xy}}{2\sqrt{z}} \right\rangle$$

$$\nabla f(3,2,6) = \left\langle \frac{\sqrt{12}}{2\sqrt{3}}, \frac{\sqrt{18}}{2\sqrt{2}}, \frac{\sqrt{6}}{2\sqrt{6}} \right\rangle$$

$$\|\vec{v}\| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\hat{v} = \left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\nabla f(3,2,6) \cdot \hat{v}$$

$$= \frac{\sqrt{12}}{2\sqrt{3}}\left(-\frac{1}{3}\right) + \frac{\sqrt{18}}{2\sqrt{2}}\left(\frac{-2}{3}\right) + \frac{\sqrt{6}}{2\sqrt{6}}\left(\frac{2}{3}\right)$$

$$= -\frac{1}{3} + \frac{3}{2}\left(\frac{-2}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)$$

$$= -\frac{1}{3} - 1 + \frac{1}{3} = -1.$$

Significance of ∇f

Thm 13:

Assume $\nabla f(\vec{x}) \neq \vec{0}$. Then $\nabla f(\vec{x})$ points in the direction along which f is increasing the fastest.

proof. If $\hat{v}$ is a unit vector, then

$$\nabla f(\vec{x}) \cdot \vec{v} = \|\nabla f(\vec{x})\| \cdot \|\vec{v}\| \cdot \cos\theta$$
$$\nabla f(\vec{x}) \cdot \vec{v} = \|\nabla f(\vec{x})\| \cdot \cos\theta$$

The dot product is largest when $\cos\theta = 1$ or $\theta = 0$.

When $\hat{v}$ & $\nabla f(\vec{x})$ are parallel; i.e, point in the same direction.

$\square$

To move the direction of fastest ascent follow $\nabla f(\vec{x})$.

Similarly, to move in a direction which f decreases the fastest follow $-\nabla f(\vec{x})$.

Ex 21 Let $T(x,y,z) = \dfrac{80}{1+x^2+2y^2+3z^2}$

be the temperature (in °C) at (x,y,z) in meters.

a.) In what direction does the temperature increase the fastest at $(1,1,-2)$

b.) What is the maximum rate of increase?

$$\nabla T = \left\langle \frac{-160x}{(1+x^2+2y^2+3z^2)^2}, \frac{-320y}{(1+x^2+2y^2+3z^2)^2}, \frac{-480z}{(1+x^2+2y^2+3z^2)^2} \right\rangle$$

$$= \frac{160}{(1+x^2+2y^2+3z^2)^2} \langle -x, -2y, -3z \rangle$$

$$\nabla T(1,1,2) = \frac{160}{256} \langle -1, -2, 6 \rangle$$

Direction only so in $\langle -1, -2, 6 \rangle$

or $\left\langle \frac{-1}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{6}{\sqrt{41}} \right\rangle$.

b.) $\|\nabla f(1,1,2)\| = \left\| \frac{160}{256} \langle -1, -2, 6 \rangle \right\|$

$$= \frac{5}{8}\sqrt{41} \approx 4° \frac{C}{m}$$

How does ∇f relate to level surfaces?

Thm 14:

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and let (x_0, y_0, z_0) lie on the level surface S defined by $f(x,y,z) = K$, K is a constant. Then $\nabla f(x_0, y_0, z_0)$ is normal (perpendicular) to the level surface. That is, if $\vec{v}$ is the tangent vector at $t=0$ of a path $\vec{c}(t)$ in S with $\vec{c}(0) = (x_0, y_0, z_0)$, then

$$\nabla f(x_0, y_0, z_0) \cdot \vec{v} = 0.$$

Defn: Tangent plane to level Surface

Let S be the surface consisting of those (x,y,z) such that $f(x,y,z) = K$, K is a constant. The tangent plane of S at a point (x_0, y_0, z_0) of S is defined by

$$\nabla f(x_0, y_0, z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

if $\nabla f(x_0, y_0, z_0) \neq \vec{0}$.

Ex 3 Find the equation of the tangent plane to the surface $xyz^2 = 6$ at $(3,2,1)$

$f(x,y,z) = xyz^2$ ($\llcorner$ 6)

$\nabla f = \langle yz^2, xz^2, 2xyz \rangle$

$\nabla f(3,2,1) = \langle 2, 3, 12 \rangle$

So $\langle 2,3,12 \rangle \cdot \langle x-3, y-2, z-1 \rangle = 0$

$2(x-3) + 3(y-2) + 12(z-1) = 0$

$2x + 3y + 12z = 24$

Ex 4 | Find a unit vector normal to the surface $x^4 + y^4 + z^4 = 3x^2y^2z^2$ at $(1,1,1)$

Consider $f(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$

The level set for $c=0$ is the desired surface.

The gradient is normal to the surface, so

$\nabla f = \langle 4x^3 - 6xy^2z^2,\ 4y^3 - 6x^2yz^2,\ 4z^3 - 6x^2y^2z \rangle$

$\nabla f(1,1,1) = \langle -2, -2, -2 \rangle$

So $\langle -2, -2, -2 \rangle$ is normal to the surface.

Normalize it $\left\langle \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}} \right\rangle = \hat{n}.$