§3.1 Iterated Partial Derivatives

In single variable calculus we could find 2nd, 3rd, 4th, ... derivatives
So $f(x) \to f'(x) \to f''(x) \to f'''(x) \to \cdots$
or

$$f(x) \longrightarrow \frac{df}{dx} \longrightarrow \frac{d^2 f}{dx^2} \longrightarrow \frac{d^3 f}{dx^3}$$

$$\frac{d}{dx}\big(f(x)\big)$$

So we had continuous functions, or better continuously differentiable functions.
[Differentiable $\Longrightarrow$ continuity]

So we have different classes of functions
C^0 = continuous functions
C^1 = differentiable continuous $(f' \in C^0)$
C^2 = 2nd differentiable continuous
↳ f'' exists & continuous.
$(f'' \in C^0)$

Defn: Class C^1
Let $f: \mathbb{R}^3 \to \mathbb{R}$ be of class C^1.
This means $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ exist

and are continuous.

What do higher order Derivatives look like?

$f(x, y, z)$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x \partial z}$$

Etc...

If $f: \mathbb{R}^2 \to \mathbb{R}$ then we have four

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial y \underset{\uparrow \text{first}}{\partial x}}, \quad \frac{\partial^2 f}{\partial x \underset{\uparrow \text{first}}{\partial y}}$$

$$f_{\underset{\uparrow \text{first}}{x}x} \qquad f_{\underset{\uparrow \text{first}}{y}y} \qquad f_{\underset{\uparrow \text{first}}{x}y} \qquad f_{\underset{\uparrow \text{first}}{y}x}$$

<u>Defn:</u> Class C^2

If $f: \mathbb{R}^3 \to \mathbb{R}$ and all the <u>2nd</u> order partials exist and are continuous, we say f is C^2.

Likewise for C^3, C^4, ...

Ex 1) Find all second partials of
$f(x,y) = x^3 + x^2y^3 - 2y^2$

$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3$ $\frac{\partial f}{\partial y} = 3x^2y^2 - 4y$

$\frac{\partial^2 f}{\partial x^2} = 6x + 2y^3$ $\frac{\partial^2 f}{\partial y^2} = 6x^2y - 4$

$\frac{\partial^2 f}{\partial y\,\partial x} = 6xy^2$ $\frac{\partial^2 f}{\partial x\,\partial y} = 6xy^2$

Ex 2) Find all second partials of
$f(x,y,z) = 2e^{xy} + z\sin(x)$

$f_x = 2ye^{xy} + z\cos(x)$, $f_y = 2xe^{xy}$, $f_z = \sin(x)$

$f_{xx} = 2y^2e^{xy} - z\sin(x)$, $f_{yy} = 2x^2e^{xy}$, $f_{zz} = 0$

$f_{xy} = 2e^{xy} + 2xye^{xy}$, $f_{yx} = 2e^{xy} + 2xye^{xy}$ $f_{zx} = \cos(x)$

$f_{xz} = \cos(x)$, $f_{yz} = 0$ $f_{zy} = 0$

Notice anything?

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

Thm 1: Clairaut's Theorem
If $f(x,y)$ is of class C^2 (twice continuously differentiable), then the mixed partial derivatives are equal; that is

$$f_{yx} = \boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}} = f_{xy}$$

Partial Differential Equations (PDEs)
We can have equations where the unknown is a function! But we have a relation with it's derivative
E.g. $y' + \frac{1}{x} y = x \sin(2x)$
(Take Math 46 for more on that)

We can similarly have equations relating partial derivatives.

↳ Wave, Schrödinger, Maxwell's

Ex3) Verify that $u(x,t) = e^{-\alpha^2 K^2 t} \sin(Kx)$ is a solution to the Heat Conduction Equation: $u_t = \alpha^2 u_{xx}$.
(α, K are constants)

$$u_t = -\alpha^2 K^2 e^{-\alpha^2 K^2 t} \sin(Kx)$$

$$u_x = K e^{-\alpha^2 K^2 t} \cos(Kx)$$

$$u_{xx} = -K^2 e^{-\alpha^2 K^2 t} \sin(Kx)$$

$$u_t \stackrel{?}{=} \alpha^2 u_{xx}$$

$$-\alpha^2 K^2 e^{-\alpha^2 K^2 t} \sin(Kx) = \alpha^2\left(-K^2 e^{-\alpha^2 K^2 t} \sin(Kx)\right)$$

✓

u is a solution to $u_t = \alpha^2 u_{xx}$.

§ 3.2 Taylor's Theorem

Given a function we can approximate it, using polynomials. In single variable:
Taylor Series

$$f(x) = \underbrace{f(a) + f'(a)(x-a)}_{\text{tangent line}} + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$+ R_n$$

where $R_n = \displaystyle\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$

<u>Example</u>

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

Or, in a different notation: $h = (x - x_0)$

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}h^k$$

$$+ R_k(x_0, h)$$

Special Case: $n=2$

Formula $f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + h_1 \frac{\partial f}{\partial x}(\vec{x}_0) + h_2 \frac{\partial f}{\partial y}(\vec{x}_0)$

$$+ \frac{1}{2} h_1^2 \frac{\partial^2 f}{\partial x^2}(\vec{x}_0) + \frac{1}{2} h_2^2 \frac{\partial^2 f}{\partial y^2}(\vec{x}_0) + 1\, h_1 h_2 \frac{\partial^2 f}{\partial x \partial y}(\vec{x}_0)$$

$$+ R_2(\vec{x}_0, \vec{h})$$

Or,

Approx

$$f(x,y) = f(x_0,y_0) + (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0)$$

$$+ \frac{1}{2}(x-x_0)^2 \frac{\partial^2 f}{\partial x^2}(x_0,y_0) + \frac{1}{2}(y-y_0)^2 \frac{\partial^2 f}{\partial y^2}(x_0,y_0)$$

$$+ \; 1\; (x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) + R_2(\vec{x}_0,\vec{h})$$

Notice: $\frac{1}{2} h_1 h_2 \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{2} h_2 h_1 \frac{\partial^2 f}{\partial y \partial x}$

So, they add.

Ex2] Compute the second-order Taylor formula and approximation for $f(x,y) = e^{-3xy}$ at $(0,0)$.

$f(0,0) = 1$

$f_x = -3ye^{-3xy}$ $\qquad$ $f_y = -3xe^{-3xy}$

$f_{xx} = 9y^2 e^{-3xy}$, $\quad f_{yy} = 9x^2 e^{-3xy}$, $\quad f_{xy} = -3e^{-3xy} + 9xye^{-3xy}$

$f_x(0,0) = 0$ $\qquad$ $f_y(0,0) = 0$

$f_{xx}(0,0) = 0$ $\qquad$ $f_{yy}(0,0) = 0$

$f_{xy}(0,0) = -3$

So, $f(\vec{h}) = 1 + -3h_1 h_2 + R_2(\vec{x_0}, h)$

Or,

$$f(x,y) = 1 - 3xy + R_2(\vec{x_0}, h)$$

Recall

$$\boxed{e^{-t} = 1 - t} + \frac{t^2}{2} - \cdots$$

§ 3.3 Extrema of Real-Valued Functions

Single-Variable Review:

Ex 1] Find the absolute and relative maximum and minimum of $f(x,y) = -3x^3 + 12x^2 - 15x + 6$ on $[0,3]$.

$= -3(x-1)^2(x-2)$

1st: First find f' to get critical numbers

$$f'(x) = -9x^2 + 24x - 15$$
$$= -3(3x^2 - 8x + 5)$$

$$0 = 3x^2 - 8x + 5 = (x-1)(3x-5)$$

$x = 1$ & $x = \frac{5}{3}$

are the critical numbers.

↳ Min or max could occur here.

$f(1) = 0$ & $f(\frac{5}{3}) = \frac{4}{9}$

2. Determine extrema

Derivative tests:

1st Derivative Test

x	0	1	4/3	5/3	2
f'	$-$	0	$+$	0	$-$

DEC ↘ Inc ↗ ↘ Dec

min max

Or,

$f''(x) = -18x + 24$

$f''(1) = +$ CCU

$f''(\frac{5}{3}) = -$ CCD

local = relative

$f(1) = 0$ is a relative min value

$f(\frac{5}{3}) = \frac{4}{9}$ is a relative max value.

3. Check end points $x = 0$, $x = 3$

So, $f(0) = 6$ absolute max

$f(3) = -12$ absolute min

(Absolute = global)

Defn:
If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is a given scalar function, a point $\vec{x_0} \in U$ is called a local/relative minimum of f if "near" $\vec{x_0}$, $f(\vec{x}) \geq f(\vec{x_0})$. Similarly, $\vec{x_0} \in U$ is a local/relative maximum of f if "near" $\vec{x_0}$ $f(\vec{x}) \leq f(\vec{x_0})$. A point $\vec{x_0}$ is a critical point of f if either
1. f is not differentiable at $\vec{x_0}$ or
2. $Df(\vec{x_0}) = 0$.

Three Options:
Local max, Local min, or saddle point.

Thm 4:
(If $U \subset \mathbb{R}^n$ is open.) Let the function $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable, and $f(\vec{x_0})$ is a local min or max or saddle point then
$Df(\vec{x_0}) = \vec{0}$. That is $\vec{x_0}$ is a critical point of f.

Special Case: $n = 2$ $f(x,y)$

$$\frac{\partial f}{\partial x}(a,b) = 0 \text{ and } \frac{\partial f}{\partial y}(a,b) = 0$$

means (a,b) is a critical point.

Ex 11 Find the local maxima/minima of $f(x,y) = x^2 + y^2$

$\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$

So $\begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \Rightarrow x = y = 0$

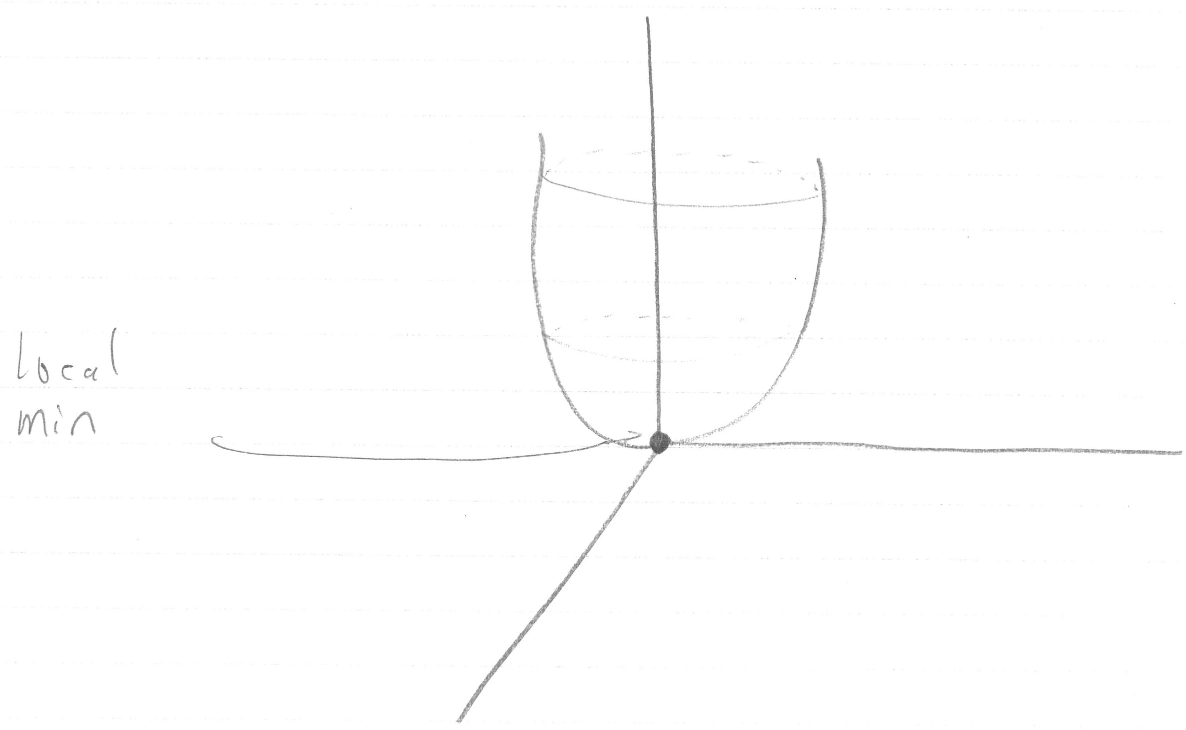
So $(0,0)$ is a critical point

$f(0,0)$ is a local minimum since $f(0,0) \leq f(x,y)$ for all x, y

Since $(0,0)$ is the only critical point, there are no local maxima.

[$f(0,0) = 0$ is also an absolute minimum]

"We're at the bottom of the bowl."



Ex2| Min/max of $f(x,y) = x^2 - y^2$?

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y$$

So $\begin{cases} 2x = 0 \\ -2y = 0 \end{cases} \Rightarrow (0,0)$ is a critical point.

But $f(0,0) = 0$ is not a local max or min.

Since $f(x,0) = -y^2 \leq 0$
and $f(0,y) = x^2 \geq 0$

$(0,0)$ is a saddle point.

Ex3| Find all the critical points of
$f(x,y) = x^4 + y^4 - 4xy + 1$

$$\frac{\partial f}{\partial x} = 4x^3 - 4y \qquad \frac{\partial f}{\partial y} = 4y^3 - 4x$$

$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases}$ $\quad y = x^3$

$\Rightarrow x^9 - x = 0$

$x(x^8 - 1) = 0$

$x(x^4 - 1)(x^4 + 1) = 0$

$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$

$x = 0, -1, 1$

So, when $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$x = -1 \Rightarrow y = -1$

So $(-1,-1)$, $(0,0)$ & $(1,1)$ are critical points of $f(x,y)$.

Ex 4 | What are the critical points of
$f(x,y) = 5(x^2+y^2)e^{-x^2-y^2}$

$$f_x = 5\left[(2x)e^{-x^2-y^2} - 2x e^{-x^2-y^2}(x^2+y^2)\right]$$
$$= 10x e^{-x^2-y^2}\left[1-(x^2+y^2)\right]$$

$$f_y = 10y e^{-x^2-y^2}\left[1-(x^2+y^2)\right]$$

$$\begin{cases} 10x e^{-x^2-y^2}\left[1-(x^2+y^2)\right] = 0 \\ 10y e^{-x^2-y^2}\left[1-(x^2+y^2)\right] = 0 \end{cases}$$

When $x = y = 0$ or $1-(x^2+y^2) = 0$
$\Leftrightarrow x^2+y^2 = 1$
a circle.

"Volcano" Function
min at bottom of crater
max along edge.

"Crazy Pig Test" – Mike's Multivariable Calculus Professor

<u>Thm 6:</u> Second-Derivative Test (n=2)

Let $f(x,y)$ be of class C^2 (on an open set $U \subset \mathbb{R}^n$). Let (x_0, y_0) be a critical point of $f(x,y)$. That is, $\frac{\partial f}{\partial x}(x_0,y_0) = 0$ & $\frac{\partial f}{\partial y}(x_0,y_0) = 0$.

Define $D = D(x_0,y_0) = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$

then

1. If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then $f(x_0,y_0)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then $f(x_0,y_0)$ is a local maximum.
3. If $D < 0$, then $f(x_0,y_0)$ is a saddle point; that is, neither a max or min.
4. $D = 0$. The test is inconclusive.

D is a function of (x,y)!

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = D^2 f = D(\nabla f) = D\langle f_x, f_y \rangle$$

↑ Determinant of Hessian

<u>Ex 1 cont</u> Classify the critical points of

$f(x,y) = x^2 + y^2$

$f_x = 2x = 0$ $\Rightarrow$ $(0,0)$ is the CP

$f_y = 2y = 0$

$f_{xx} = \underset{>0}{2}$, $f_{xy} = 0$, $f_{yy} = 2$

So $D(x,y) = (2)(2) - (0)^2 = 4$

$D(0,0) = 4 > 0$

Thus $(0,0)$ is where a local minimum occurs.

↳ $f(0,0) = 0$.

Ex 2 cont] Classify the critical points of
$f(x,y) = x^2 - y^2$

$$\begin{cases} f_x = 2x = 0 \\ f_y = -2y = 0 \end{cases} \Rightarrow (0,0) \text{ is a C.P.}$$

$f_{xx} = 2 \qquad f_{yy} = -2 \qquad f_{xy} = 0$

$D(x,y) = 2(-2) - (0)^2 = -4$

$D < 0$

Thus $(0,0)$ is a saddle point.

Ex 3 cont] Classify the critical points of
$f(x,y) = x^4 + y^4 - 4xy + 1$

$$\begin{cases} f_x = 4x^3 - 4y = 0 \\ f_y = 4y^3 - 4x = 0 \end{cases} \Rightarrow (0,0),\ (1,1),\ (-1,-1)$$

$f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = -4$

$D(x,y) = (12x^2)(12y^2) - (-4)^2 = 144x^2y^2 - 16$

$D(0,0) = -16 < 0 \Rightarrow$ Saddle point.

$D(1,1) = 144 - 16 = 128 > 0$, $f_{xx}(1,1) = 12 > 0$

$\hookrightarrow (1,1)$ is where a local minimom occurs.

$D(-1,-1) = 128 > 0$, $f_{xx}(-1,-1) = 12 > 0$

$\hookrightarrow (-1,-1)$ is where a local minimom occurs.

$f(1,1) = -1 = f(-1,-1)$

$\uparrow$ local minimom value.

Ex 5) Find and classify the critical points of $f(x,y) = (x-y)(1-xy)$

$$= x - x^2y - y + xy^2$$

$$\begin{cases} f_x = 1 - 2xy + y^2 = 0 \\ f_y = -x^2 - 1 + 2xy = 0 \end{cases}$$

Add them

$$-x^2 + y^2 = 0 \implies y^2 = x^2 \implies y = \pm x$$

Plug into top, both versions

$$1 - 2x^2 + x^2 = 0 \longrightarrow x^2 = 1 \quad x = \pm 1$$

$$1 + 2x^2 + x^2 = 0 \longrightarrow x^2 = -\frac{1}{3} \quad \text{\#}$$

So $y = +x$

$(1,1)$ & $(-1,-1)$ are my critical points

$f_{xx} = -2y \qquad f_{yy} = 2x \qquad f_{xy} = -2x + 2y$

$$D(x,y) = (-2y)(2x) - (-2x + 2y)$$

$$= -4xy + 2x - 2y$$

$D(1,1) = -4 + 2 - 2 = -6 < 0$

$(1,1)$ is a saddle point

$D(-1,-1) = -4 - 2 + 2 = -6 < 0$

$(-1,-1)$ is a saddle point.

Thm 7: Extreme Value Theorem

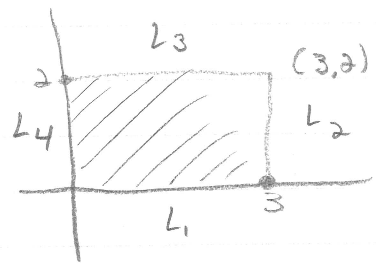
Let D be a closed and bounded set in $\mathbb{R}^n$ and let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Then f attains its absolute maximum value and absolute minimum value at some points $\vec{x}_0$ and $\vec{x}_1 \in D$.

Strategy for finding absolute max/min

Let f be C^0 on bounded set D.

1. Find the values of f at the critical points (No testing needed)
2. Find the extreme values of f on the boundary of D. (Reduced to Calc 1) (Parameterize the boundary)
3. The largest of these values is the absolute maximum value of f and the smallest is the absolute minimum of f.

Ex 6 | Find the absolute maximum and minimom values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le 2\}$



1. Locate CP

$$\begin{cases} f_x = 2x - 2y = 0 \\ f_y = -2x + 2 = 0 \end{cases}$$

Add $\quad -2y + 2 = 0$

$y = 1$

$2x - 2 = 0 \quad \Rightarrow \quad x = 1$

$(1,1)$ is the critical point.

$f(1,1) = 1 - 2 + 2 = 1$

2. Find extreme values on boundary of D.

On L_1

Parameterize as $\vec{c}_1(t) = (t, 0) \qquad 0 \le t \le 3$

So $(f \circ c_1)(t) = t^2$

Absolute min at $t = 0$, of $0 \qquad (0,0)$

Absolute max at $t = 3$, of $9 \qquad (3,0)$

On L_2: $\vec{c}_2(t) = (3, t) \qquad 0 \le t \le 2$

$(f \circ \vec{c}_2)(t) = 9 - 6t + 2t = 9 - 4t$

Absolute min at $t = 2$, of $5 \qquad (3,2)$

Absolute max at $t = 0$, of $9 \qquad (3,0)$

On L_3: $\vec{c}_3 = (t, 2) \qquad 0 \le t \le 3$

$2t - 4$ $\quad (f \circ \vec{c}_3)(t) = t^2 - 4t + 4$

$= (t - 2)^2$

Absolute min at $t = 2$ of 0, $\quad (2,2)$

Absolute max at $t = 0$ of 4, $\quad (0,2)$

Ex 6 cont.

On L_4: $\vec{c_4} = (0, t)$ $\quad 0 \le t \le 2$

$(f \circ \vec{c_4})(t) = 2t$

Absolute min at $t=0$ of 0 $\quad (0,0)$

Absolute max at $t=2$ of 4 $\quad (0,2)$

So, the absolute max is at $(3,0)$

$f(3,0) = 9$ abs. max value

and the absolute min is at $(0,0)$ and $(2,2)$

$f(0,0) = f(2,2) = 0$.

Ex 7) Find the maximum and minimum values of the function $f(x,y) = x^2 + y^2 - x - y + 1$ in the disk D defined by $x^2 + y^2 \le 1$

1. Find Critical points

$\frac{\partial f}{\partial x} = 2x - 1 = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 2y - 1 = 0$

$x = \frac{1}{2}$ $\qquad$ $y = \frac{1}{2}$

$(\frac{1}{2}, \frac{1}{2})$ $\qquad$ $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + 1 = \frac{1}{2}$

2. The Boundary is the unit circle, we can parameterize as follows

$\vec{c}(t) = (\cos(t), \sin(t))$ $\quad 0 \le t \le 2\pi$

$$(f \circ \vec{c})(t) = \underbrace{\cos^2(t) + \sin^2(t)}_{1} - \cos(t) - \sin(t) + 1$$

Ex 7 cont

So $(f \circ \vec{r})(t) = 2 - \sin(t) - \cos(t)$

$$\frac{d}{dt}(f \circ \vec{r})(t) = -\cos(t) + \sin(t) = 0$$

$$\cos(t) = \sin(t)$$
$$t = \frac{\pi}{4}, \frac{5\pi}{4}$$

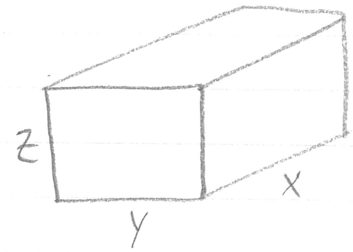
So

$$(f \circ \vec{r})\left(\frac{5\pi}{4}\right) = 2 - -\frac{\sqrt{2}}{2} - -\frac{\sqrt{2}}{2} = 2 + \sqrt{2} = f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$
$$(f \circ \vec{r})(\pi/4) = 2 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 2 - \sqrt{2} = f(\sqrt{2}/2, \sqrt{2}/2)$$
$$(f \circ \vec{r})(0) = 2 - 0 - 1 = 1 = f(0,1)$$
$$(f \circ \vec{r})(2\pi) = 2 - 0 - 1 = 1$$

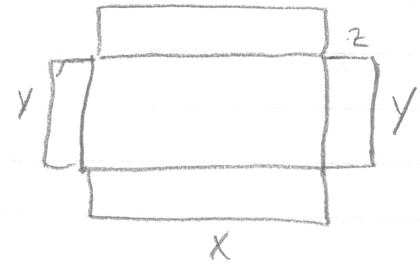
Thus the absolute maximom is $2+\sqrt{2}$ at $(-\sqrt{2}/2, -\sqrt{2}/2)$ and the absolute minimom is $\frac{1}{2}$ at $(\frac{1}{2}, \frac{1}{2})$.

Ex 81 A rectangular box without a lid is to be made from $12 \, m^2$ of metal. Find the maximum volume of this box.

$V = xyz$



But I only have so much metal



So, I have a restriction or constraint on the surface area.

$$2xz + 2yz + xy = 12 \qquad \text{(No lid)}$$

↳ Solve for z and reduce to $n = 2$

$$2z(x+y) = 12 - xy \implies z = \frac{12 - xy}{2(x+y)}$$

So

$$V(x,y) = xy\left(\frac{12 - xy}{2(x+y)}\right) = \frac{12xy - x^2y^2}{2x + 2y}$$

$$\frac{\partial V}{\partial x} = \frac{(12y - 2xy^2)(2x + 2y) - 2(12xy - x^2y^2)}{(2x + 2y)^2}$$

$$= \frac{\cancel{24xy} + 24y^2 - 4x^2y^2 - 4xy^3 - \cancel{24xy} + 2x^2y^2}{(2x + 2y)^2}$$

$$= \frac{24y^2 - 2x^2y^2 - 4xy^3}{(2x + 2y)^2} = \frac{2y^2(12 - x^2 - 2xy)}{(2x + 2y)}$$

Cont. →

Ex 8 cont

Likewise, $\frac{\partial V}{\partial y} = \frac{2x^2(12 - y^2 - 2xy)}{(2x+2y)^2} = 0$

$\frac{\partial V}{\partial x} = \frac{2y^2(12 - x^2 - 2xy)}{(2x+2y)^2} = 0$

$(0,0) \Rightarrow V(0,0) = 0$

or $\begin{cases} 12 - y^2 - 2xy = 0 \\ 12 - x^2 - 2xy = 0 \end{cases}$

Subtract

$-y^2 + x^2 = 0$ $\quad y^2 = x^2$

$y = \pm x = x$

(y must be positive)

$12 - x^2 - 2x^2 = 0$

$12 - 3x^2 = 0$

$4 = x^2$

$+2 = x \Rightarrow y = 2$

Then $z = \frac{12 - (2)(2)}{2(2+2)} = \frac{8}{8} = 1$

Cube

$0 \le x \le 12$ ← Boundary, intuitively our boundary causes issues as we're in $\mathbb{R}^3$...

$0 \le y \le 12$

$0 \le z \le 12$

So as this is a real life situation all boundary cases will net in 0 volume, as either $x=0, y=0, z=0$ or $x=12, y=12, z=12$

$\Rightarrow V = 0.$

Thus, the max Volume is $V = 2 \cdot 2 \cdot 1 = 4\,m^3$ at $(2,2,1)$.

§ 3.4 Constrained Extrema and Lagrange Multipliers

Consider $f(x,y)$ subject to $g(x,y)=K$ specific level curve of g.

Consider all level curves of $f(x,y)=c$

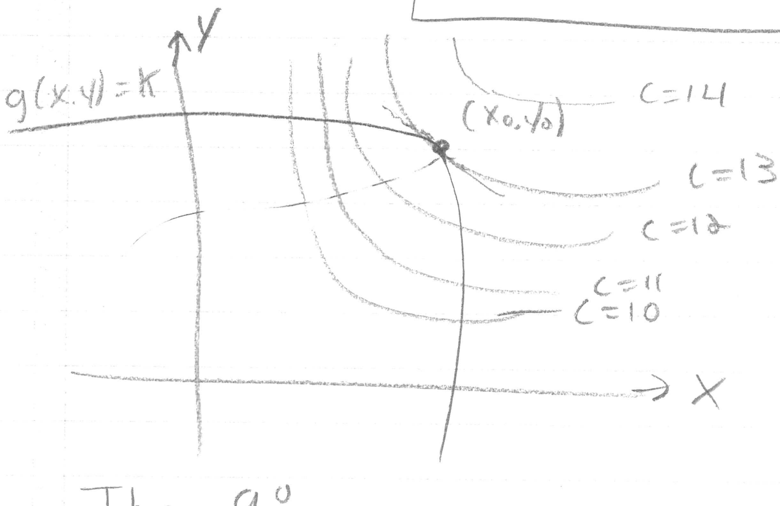
To find the local maximum, we want to find the largest c so that $g(x,y)=K$ intersects that level curve.

This occurs precisely when the curves touch at a common tangent line. (If not we could make c larger)

Since they have the same tangent line at the point of intersection, they must also have the same normal lines.

Thus the gradient vectors are parallel that is

$$\boxed{\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)} \quad (n=2)$$



λ is a Lagrange Multiplier

Thm 9:

If f, when constrained to a surface S, has a local maximum or minimum at $\vec{x}_0$, then $\nabla f(\vec{x}_0)$ is perpendicular to S at $\vec{x}_0$.

Thm 8: Method of Lagrange Multipliers $(n=3)$

To find the local maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=K$

[Assuming the extrema exist and
$\nabla g \neq \vec{0}$ on the surface $g(x,y,z)=K$]

1.) Find all values of x, y, z, λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $$g(x,y,z) = K$$

2. Evaluate f at all points (x,y,z) that result from 1. The largest is the local maximum and the smallest is the local minimum value of f.

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle f_x, f_y, f_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = K \end{cases}$$

4 Equations, 4 unknowns

(Don't need λ)

Ex 11 Find the local extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2+y^2=1$.

$g(x,y) = 1$

$\nabla f = \langle 2x, 4y \rangle$

$\nabla g = \langle 2x, 2y \rangle$

So
$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 1 \end{cases} \qquad \begin{cases} 2x = \lambda 2x & (1) \\ 4y = \lambda 2y & (2) \\ x^2+y^2 = 1 & (3) \end{cases}$$

From (1), either $x = 0$, or $\lambda = 1$.

$x = 0 \xrightarrow{\text{by } (3)} y = \pm 1$

$\lambda = 1 \xrightarrow{\text{by } (2)} y = 0 \xrightarrow{\text{by } (3)} x = \pm 1$

Thus we have
$(0, +1)$, $(0,-1)$, $(+1, 0)$, $(-1, 0)$

$f(0, \pm 1) = 0^2 + 2 \cdot 1^2 = 2$ local maximum

$f(\pm 1, 0) = 1^2 + 2(0)^2 = 1$ local minimum

Ex 2 Maximize $f(x,y,z) = x + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$

(Thm 7: EVT ensures existence)

$\nabla f = \langle 1, 0, 1 \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$$\begin{cases} 1 = \lambda 2x & (1) \\ 0 = \lambda 2y & (2) \\ 1 = \lambda 2z & (3) \\ x^2 + y^2 + z^2 = 1 & (4) \end{cases}$$

$\Rightarrow \lambda \neq 0 \quad x \neq 0$

$\Rightarrow y = 0$

$\Rightarrow \lambda \neq 0, \ z \neq 0$

From (1) and (3) we get that

$$\lambda 2x = \lambda 2z \quad \& \ \lambda \neq 0$$

$$2x = 2z$$

$$x = z$$

Plug all this into 4

$$x^2 + 0^2 + x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

$\Rightarrow \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ are candidates

So

$f\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ local max

$f\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}.$ local min.

Ex3 A rectangular box without a lid is to be made from $12m^2$ of metal. Find the maximum value of this box.

Maximize $V(x,y,z) = xyz$

subject to $\underbrace{2xz + 2yz + xy}_{g(x,y,z)} = 12$

$\nabla V = \langle yz, xz, xy \rangle$

$\nabla g = \langle 2z + y, 2z + x, 2x + 2y \rangle$

$$\begin{cases} V_x = \lambda g_x \\ V_y = \lambda g_y \\ V_z = \lambda g_z \\ 2xz + 2yz + xy = 12 \end{cases} \qquad \begin{aligned} yz &= \lambda(2z+y) && (1) \\ xz &= \lambda(2z+x) && (2) \\ xy &= \lambda(2x+2y) && (3) \\ 2xz + 2yz + xy &= 12 && (4) \end{aligned}$$

Solving systems is hard!

x (1), y (2), z (3), we get

(5) $xyz = \lambda x(2z+y)$

(6) $xyz = \lambda y(2z+x)$

(7) $xyz = \lambda z(2x+2y)$

$\lambda \neq 0$ since If $\lambda = 0$, $yz = xz = xy = 0$ then $2(0) + 2(0) + 0 \neq 12$ from (4).

From (5) & (6) then,

$\lambda x(2z+y) = \lambda y(2z+x)$ $\quad 2\lambda \neq 0$

$2xz + xy = 2yz + xy$

$xz = yz$ $\quad z \neq 0$ or else $V = 0$

$x = y$

Cont.

Ex3 cont'd $\qquad y = x$

Take ⑥ and ⑦

$$\lambda y(2z + x) = \lambda z(2x + 2y) \quad \lambda \neq 0$$
$$2yz + xy = 2xz + 2yz$$
$$xy = 2xz \qquad x \neq 0 \text{ or (else } V = 0)$$
$$y = 2z$$

So, $x = y = 2z$

Plug this into ④

$$4z^2 + 4z^2 + 4z^2 = 12$$
$$12z^2 = 12$$
$$z^2 = 1$$
$$z = \pm 1 \rightarrow z = 1$$
$$\Rightarrow x = y = 2$$

Thus $(2,2,1)$ is our candidate and we have $V(2,2,1) = 4 \ m^3$.

Lagrange Multiplier Method for Absolute Extrema

Let f be a C^1 function on a bounded set D. The boundary of D is "smooth" (C^∞) and is the level set of g with $\nabla g \neq \vec{0}$.

1. Find all critical points of f inside D, not on the boundary. (No testing needed)
2. Use the method of Lagrange multipliers to locate all the critical points of f on the boundary.
3. Compute the values of f at all these critical points.
4. Largest value is the absolute maximum and the smallest value is the absolute minimum.

Compare with Ex1

Ex 4| Find the absolute extrema of the function $f(x,y) = x^2 + 2y^2$ on the disk $x^2+y^2 \le 1$.

1. Find CP's

$f_x = 2x \qquad f_y = 4y \qquad \Rightarrow \qquad (0,0)$

2. $\begin{cases} \nabla f = \lambda \nabla g \\ x^2+y^2 = 1 \end{cases}$

$\begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2+y^2=1 \end{cases}$

$x=0 \Rightarrow y=\pm 1$

$x\neq 0 \Rightarrow \lambda = 1 \Rightarrow y=0 \Rightarrow x=\pm 1$

$(0, \pm 1)$ and $(\pm 1, 0)$

3. $f(0,\pm 1) = 2$, $f(\pm 1, 0) = 1$, $f(0,0) = 0$

4. Absolute Max $\qquad\qquad$ Absolute Min.

Ex5| Find the absolute maximum and minimum of $f(x,y,z) = x^2+y^2+z^2-x+y$ on the Ball, $D = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$

1. Find critical points. (on inside "< 1")

$f_x = 2x-1$
$f_y = 2y+1$ $\Rightarrow$ $(\frac{1}{2}, \frac{-1}{2}, 0)$
$f_z = 2z$

2. Find critical points on the boundary "= 1"

$\underbrace{x^2+y^2+z^2}_{g(x,y,z)}=1$

$$\begin{cases} 2x-1 = \lambda 2x & (1) \\ 2y+1 = \lambda 2y & (2) \\ 2z = \lambda 2z & (3) \\ x^2+y^2+z^2=1 & (4) \end{cases}$$

If $\lambda = 1 \Rightarrow -1 = 0$ ↯

So $\lambda \neq 1$

If $\lambda = 0$ then we get $(\frac{1}{2}, -\frac{1}{2}, 0)$ but $x^2+y^2+z^2 \neq 1$

So, $\lambda \neq 0$.

Since $\lambda \neq 0$, from (3) we get

$2z - \lambda 2z = 0$
$2z(1-\lambda) = 0$
$\hookrightarrow \lambda \neq 1$

So $z = 0$.

(4) becomes $\Rightarrow x^2+y^2=1$ (5)

Solve (1) and (2) for x, y now

$2x - \lambda 2x = 1$

$x = \dfrac{1}{2(1-\lambda)}$ $\qquad y = \dfrac{-1}{2(1-\lambda)}$

Plug into (5), $\dfrac{1}{4(1-\lambda)^2} + \dfrac{1}{4(1-\lambda)^2} = 1$

Ex5 cont.

$$\frac{1}{2(1-\lambda)^2} = 1 \Rightarrow \frac{1}{2} = (1-\lambda)^2$$

$$\pm\sqrt{\tfrac{1}{2}} = 1-\lambda$$

$$\lambda = 1 \mp \frac{1}{\sqrt{2}}$$

Now, $x = \dfrac{1}{2(1-(1\mp\frac{1}{\sqrt{2}}))} = \dfrac{1}{2(\pm\frac{1}{\sqrt{2}})} = \dfrac{1}{\pm\frac{2}{\sqrt{2}}} = \pm\dfrac{1}{\sqrt{2}}$

Likewise with $y = \pm\frac{1}{\sqrt{2}}$. $z=0$

Critical points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2} + \frac{1}{2} + 0 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1$

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2} + \frac{1}{2} + 0 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1 - \sqrt{2}$

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2} + \frac{1}{2} + 0 - -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1 + \sqrt{2}$ Abs max

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2} + \frac{1}{2} + 0 - -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1$

$f\left(\frac{1}{2}, -\frac{1}{2}, 0\right) = \frac{1}{4} + \frac{1}{4} + 0 - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$ Abs min