§4.2 Arc Length

$\frac{dy}{dx}$

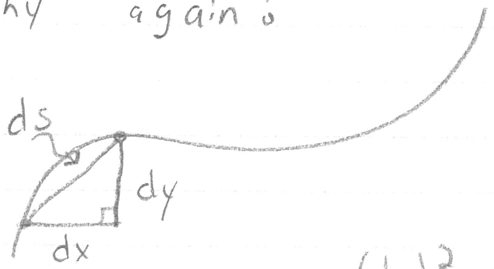
$\|$

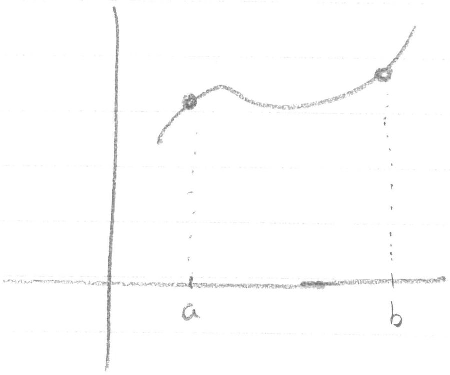
$f'(x)$

Recall from single variable calculus

$$\mathcal{L} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

Why again?



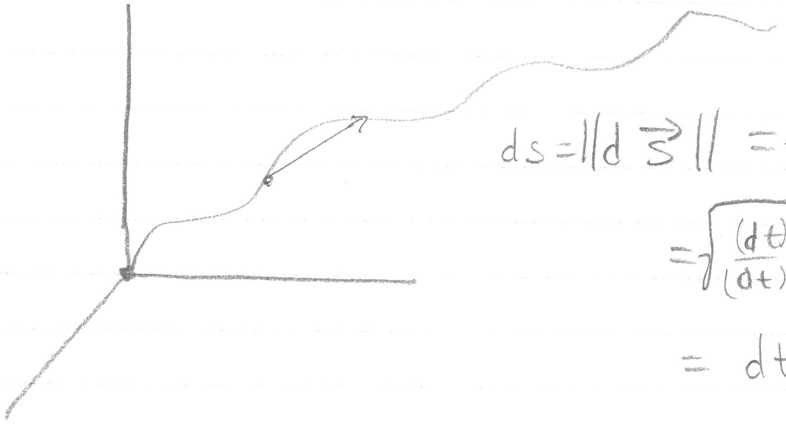


$$(dx)^2+(dy)^2=(ds)^2$$
$$\sqrt{(dx)^2+(dy)^2}=ds$$

"Add" all my tiny "ds"'s up. $\int_a^b ds$

$$\rightsquigarrow \int_a^b \sqrt{(dx)^2+(dy)^2} = \int_a^b \sqrt{(dx)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)}$$
$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

What does this look like in higher dimensions as a curve or path.



$$ds = \|d\vec{s}\| = \sqrt{(dx)^2+(dy)^2+(dz)^2}$$
$$= \sqrt{\frac{(dt)^2}{(dt)^2}\left((dx)^2+(dy)^2+(dz)^2\right)}$$
$$= dt\sqrt{\underset{\overset{\downarrow}{x'(t)}}{\left(\frac{dx}{dt}\right)^2}+\underset{\overset{\downarrow}{y'(t)}}{\left(\frac{dy}{dt}\right)^2}+\underset{\overset{\downarrow}{z'(t)}}{\left(\frac{dz}{dt}\right)^2}}$$

Defn: Arc-Length differential

An infinitesimal displacement of a particle following a path $\vec{c}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ is $d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} = \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right) dt$

and its length

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

is the differential of arc length.

Defn: Arc Length $(n=3)$

The length of the path $\vec{c}(t) = (x(t), y(t), z(t))$ for $t_0 \le t \le t_1$ is:

$$\mathcal{L}(\vec{c}(t)) = \int_{t_0}^{t} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ dt$$

$$= \int_{t_0}^{t} \|\vec{c}\,'(t)\|\ dt$$

Ex1| What is the arc length of the circle with radius 5.

$$\vec{c}(t) = (5\cos(t), 5\sin(t)) \qquad 0 \leq t \leq 2\pi$$

Intuition: what's the circumference? $2\pi r$

$\hookrightarrow$ $\underline{10\pi}$

$$\vec{c}\,'(t) = (-5\sin(t), 5\cos(t))$$

$$\ell = \int_0^{2\pi} \|\vec{c}\,'(t)\|\, dt = \int_0^{2\pi} \sqrt{25\sin^2(t) + 25\cos^2(t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{25}\, dt = 5 \cdot 2\pi = 10\pi \quad \checkmark$$

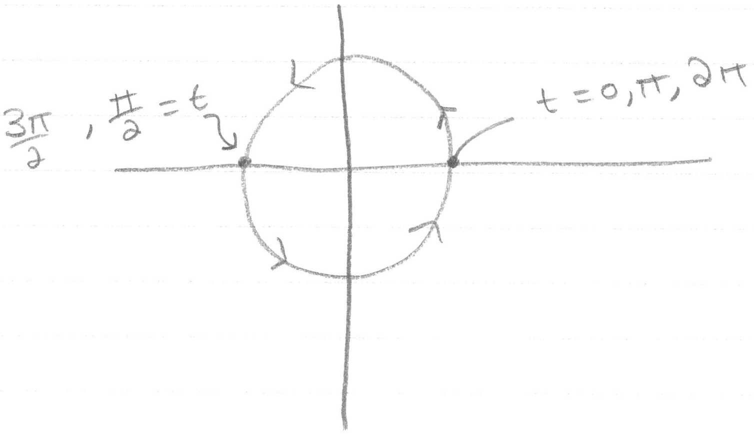
But $\vec{c}(t) = (5\cos(2t), 5\sin(2t))$ $\quad 0 \leq t \leq 2\pi$
also is a circle with $r = 5$...

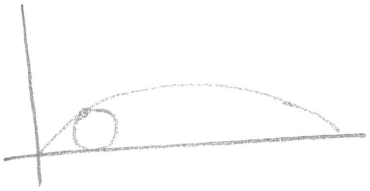
$$\vec{c}\,'(t) = (-10\sin(2t), 10\cos(2t))$$

$$\ell = \int_0^{2\pi} \sqrt{100\sin^2(2t) + 100\cos^2(2t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{100}\, dt = 10 \cdot 2\pi = 20\pi$$

What happened?
We went around twice the 2nd time!





Ex 2 Find the arc length of one arch of the cycloid. $\vec{r}(t) = (t - \sin(t),\ 1 - \cos(t))$

$0 \leq t \leq 2\pi$

$$\vec{r}\,'(t) = (1 - \cos(t),\ \sin(t))$$

$$\|\vec{r}\,'(t)\| = \sqrt{(1-\cos(t))^2 + \sin^2(t)}$$

$$= \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}$$

$$= \sqrt{2 - 2\cos(t)}$$

Half - Angle

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$$

$$\mathcal{L} = \int_0^{2\pi} \sqrt{2(1-\cos(t))}\ dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)}\ dt$$

$$= \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt = -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi}$$

$$= -4\cos(\pi) - -4\cos(0)$$

$$= 4 + 4 = 8$$

Ex3| Find the arc length of

$$\vec{c}(t) = (1, 3t^2, t^3) \qquad 0 \le t \le 1$$

$$\vec{c}\,'(t) = (0, 6t, 3t^2)$$

$$\mathcal{L} = \int_0^1 \sqrt{0^2 + 36t^2 + 9t^4}\, dt$$

$$= \int_0^1 \sqrt{9t^2(4+t^2)}\, dt = \int_0^1 3t\sqrt{4+t^2}\, dt$$

$u = 4 + t^2$

$du = 2t\, dt$

$\frac{1}{2} du = t\, dt$

$$= \int_\square^\square \frac{3}{2} u^{\frac{1}{2}}\, du$$

$$= \frac{3}{2}\frac{2}{3} u^{3/2} \Big|_\square^\square = (4+t^2)^{3/2} \Big|_0^1$$

$$= 5^{3/2} - 4^{3/2}$$

$$= 5\sqrt{5} - 8.$$

Ex 4| Find the arc length of

$\vec{r}(t) = (\ln(\sqrt{t}), \sqrt{3}\,t, \frac{3}{2}t^2)$ $\quad 1 \le t \le 2$

$\vec{r}\,'(t) = \left(\frac{1}{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}}, \sqrt{3}, 3t\right)$

$$L = \int_1^2 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$= \int_1^2 \sqrt{\frac{1}{4}\frac{1}{t^2} + 3 + 9t^2}\, dt$$

Find Common Denominator

$$= \int_1^2 \sqrt{\frac{1 + 12t^2 + 36t^4}{4t^2}}\, dt$$

$$= \int_1^2 \sqrt{\frac{(1+6t^2)^2}{(2t)^2}}\, dt = \int_1^2 \frac{1+6t^2}{2t}\, dt.$$

$$= \int_1^2 \frac{1}{2}\frac{1}{t} + 3t\,dt = \frac{1}{2}\ln(t) + \frac{3}{2}t^2\Big|_1^2$$

$$= \left(\frac{1}{2}\ln(2) + \frac{3}{2}\cdot 4\right) - \left(0 + \frac{3}{2}\right)$$

$$= \frac{1}{2}\ln(2) + \frac{9}{2}$$

§4.3 Vector Field

We been dealing with functions $f: \mathbb{R}^{\overset{1,2,3}{n}} \longrightarrow \mathbb{R}^{\overset{1,2,3}{m}}$.

$f(x_1, x_2, \dots x_n) = (f_1(x_1, \dots x_n), f_2(x_1, \dots x_n), \dots, f_m(x_1, \dots x_n))$

Input n variable get out m outputs. We been blurring this line between point and vector a lot... We now will consider functions that output vectors. Each point on the plane (x,y) or in space (x,y,z) will have a corresponding vector

__Defn:__ Vector Field

A vector field in $\mathbb{R}^n$ is a map $\vec{F}: A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $\vec{x}$ in its domain A a vector $\vec{F}(\vec{x})$.

If $n=2$, $\vec{F}$ is called a vector field in the plane.

If $n=3$, $\vec{F}$ is called a vector field in space.

Compared with a __scalar__ field $f: A \subset \mathbb{R}^n \to \mathbb{R}$, which assigns a number to each point.

In this class, we've already been working with one, the gradient vector field.

$$\nabla f = \langle \underset{\uparrow}{f_x}, \underset{\uparrow}{f_y}, \underset{\uparrow}{f_z} \rangle$$

Component Scalar Fields.

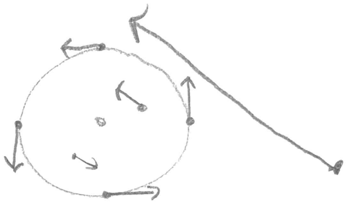
__Examples in real life:__

Velocity of water in pipe

Magentic Field

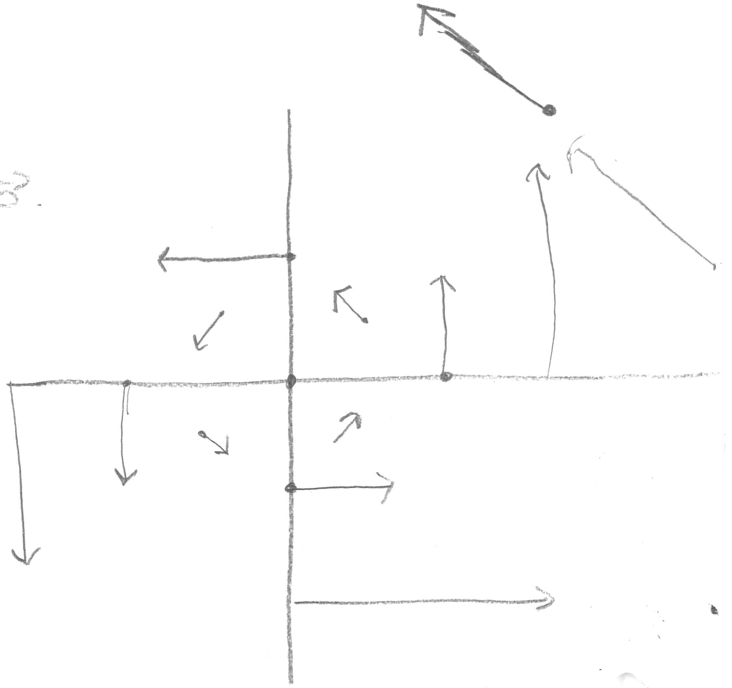
Electric Field

Or, perhaps a ....



Ex 1| Record player / Merry-Go Round / Carousel

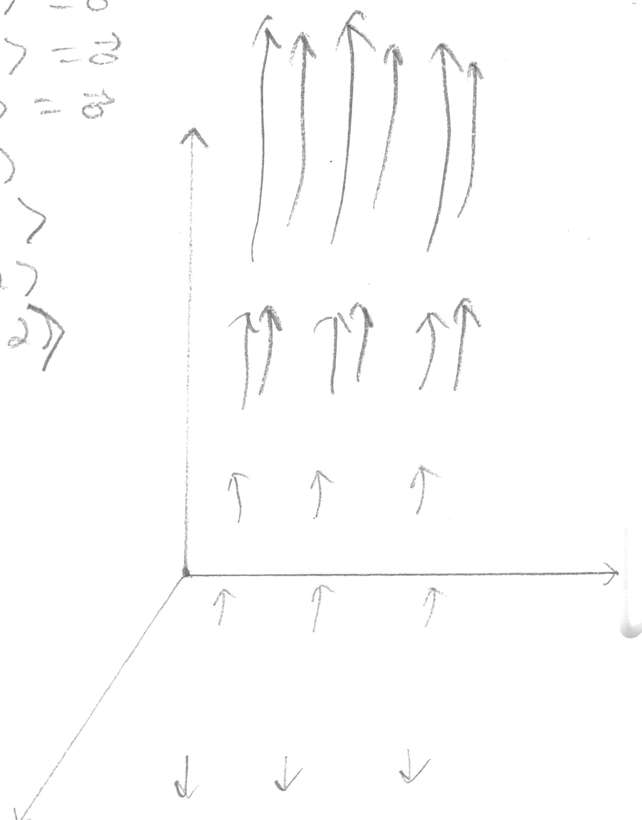
$\vec{F}(x,y) = -y\,\hat{i} + x\,\hat{j}$



(x,y)	$\langle -y, x \rangle$
$(0,0)$	$\langle 0, 0 \rangle = \vec{0}$
$(1,0)$	$\langle 0, 1 \rangle$
$(0,1)$	$\langle -1, 0 \rangle$
$(-1,0)$	$\langle 0, -1 \rangle$
$(0,-1)$	$\langle 1, 0 \rangle$
$(\frac{1}{2}, \frac{1}{2})$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
$(2,2)$	$\langle -2, 2 \rangle$
$(1,3)$	$\langle -3, 1 \rangle$
$(3,1)$	$\langle -1 \quad 3 \rangle$

Ex 2| $\vec{F}(x,y,z) = \langle 0, 0, z \rangle = z\hat{k}$

(x,y,z)	$\langle 0, 0, z \rangle$
$(0,0,0)$	$\langle 0, 0, 0 \rangle = \vec{0}$
$(1,0,0)$	$\langle 0, 0, 0 \rangle = \vec{0}$
$(1,1,0)$	$\langle 0, 0, 0 \rangle = \vec{0}$
$(1,1,1)$	$\langle 0, 0, 1 \rangle$
$(1,0,1)$	$\langle 0, 0, 1 \rangle$
$(3,-4,2)$	$\langle 0, 0, 2 \rangle$
$(1,0,-2)$	$\langle 0, 0, -2 \rangle$



Gradient Vector Fields

Recall: Grad(f), ∇f

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)\,\hat{i} + \frac{\partial f}{\partial y}(x,y,z)\,\hat{j} + \frac{\partial f}{\partial z}(x,y,z)\,\hat{k}$$

At each point (x,y,z), $\nabla f(x,y,z)$ points in the direction along which f is increasing the fastest.

Ex3) Find the gradient vector field of
$f(x,y) = x^2y - y^3$

$$\nabla f(x,y) = \langle 2xy,\ x^2 - 3y^2\rangle = 2xy\,\hat{i} + (x^2-3y^2)\,\hat{j}$$

∇f $\perp$ to level sets

Ex4) Gravity: Newton's Law of Gravitation states that the magnitude of the gravitional force between two objects with masses m & M is $\|\vec{F}\| = \frac{mMG}{r^2}$ and the force is

$$\vec{F} = \frac{-mMG}{r^3}\vec{r} \quad \text{where } \vec{r}(x,y,z) = (x,y,z), \ \|\vec{r}\| = r.$$

This force points towards the object with more mass.

This (force) vector field is rather special since if $V = \frac{-mMG}{r} = \frac{-mMG}{\sqrt{x^2+y^2+z^2}}$

then
$$-\nabla V = \left\langle \frac{-mMG}{(x^2+y^2+z^2)^{3/2}}x,\ \frac{-mMG\,y}{(x^2+y^2+z^2)^{3/2}},\ \frac{-mMG\,z}{(x^2+y^2+z^2)^{3/2}}\right\rangle$$

$$= \frac{-mMG}{r^2}\left\langle \frac{x}{\|\vec{r}\|}, \frac{y}{\|\vec{r}\|}, \frac{z}{\|\vec{r}\|}\right\rangle = \frac{-mMG}{r^3}\vec{r}.$$

Defn: Conservative Vector Field

A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some function f. That is, $\nabla f = \vec{F}$. f is a potential function for $\vec{F}$.

↳ Eg. Gravity

Ex5| Show that the vector field $\vec{F} = \langle y, -x \rangle$ is not a gradient vector field.

So if there is an f, then $y, -x \in C^1$

$$f_x = y \longrightarrow f_{xy} = 1 \quad \curvearrowleft \neq$$
$$f_y = -x \longrightarrow f_{xy} = -1$$

But $1 \neq -1$.

Thus f can't exists.

Defn: Flow Line

If $\vec{F}$ is a vector field, a flow line for $\vec{F}$ is a path $\vec{c}(t)$ such that

$$\vec{c}\,'(t) = \vec{F}(c(t)).$$

Ex6| Show that $\vec{c}(t) = (\cos(t), \sin(t))$ is a flow line for $\vec{F}(x,y) = \langle -y, x \rangle$

$$\vec{c}\,'(t) = \langle -\sin(t), \cos(t) \rangle = \langle -\sin(t), \cos(t) \rangle$$
$$= \vec{F}(c(t))$$

✓

§4.4 Divergence and Curl

Before we've used ∇f to represent the gradient of a scalar valued function, that is $\nabla f = \langle f_x, f_y, f_z \rangle = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

∇f, takes a scalar and produces a vector.
We explore ∇ in other ways.
(Del, Nabla)

Defn We will call it the <u>del</u> <u>operator</u>

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

In single variable calculus, we had $\frac{d}{dx}$ operating on $f(x)$ to produce $\frac{df}{dx} = f'(x)$.

<u>Defn</u>: Divergence (of a vector field)
If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, the divergence of $\vec{F}$ is the scalar (field)

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

(works in $\mathbb{R}^n$)

<u>Ex 11</u> If $F = xz\hat{i} + xyz\hat{j} - y^2\hat{k}$, find $\text{div}(\vec{F})$.

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = z + xz - 0 = z + xz$$

What does divergence mean physically?
If we imagine $\vec{F}$ to represent the velocity of a fluid, divergence represents the rate of expansion per unit volume* under the flow of fluid. If $\text{div}(\vec{F}) = 0$, the fluid is incompressible.
If $\text{div}(\vec{F}) < 0$, the fluid is compressing.
If $\text{div}(\vec{F}) > 0$, the fluid is expanding. (Diverging)

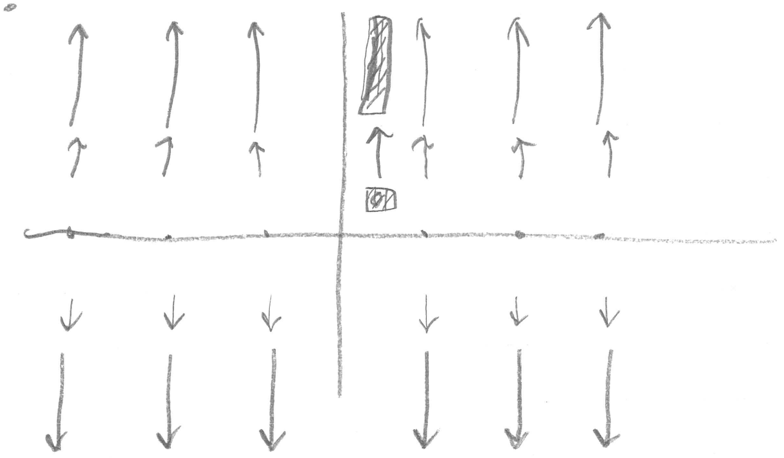
(*Volume in $\mathbb{R}^3$, Area in $\mathbb{R}^2$)

Div

Ex2 Consider $\vec{F}(x,y) = \langle 0, y \rangle = y\hat{j}$

$\text{div}(\vec{F}) = 1.$

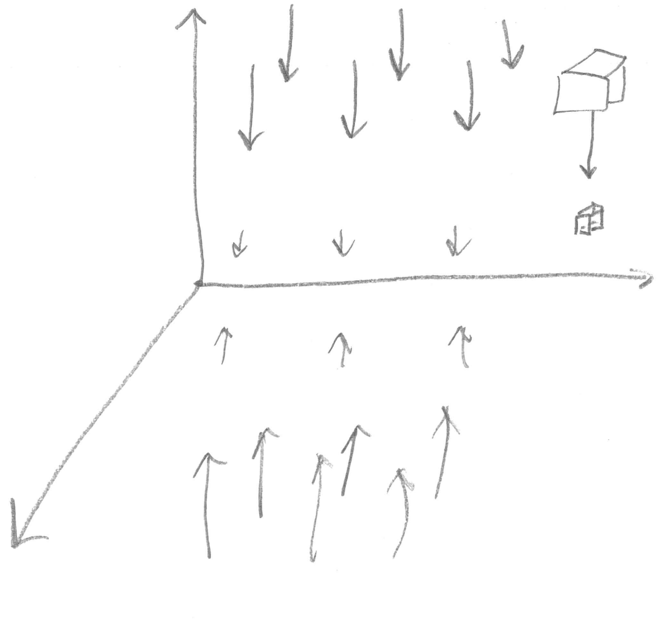
Fluid Expands



Ex3 Consider $\vec{F}(x,y,z) = -z\hat{k} = \langle 0, 0, -z \rangle$

$\text{div}(\vec{F}) = -1$

Fluid compresses.

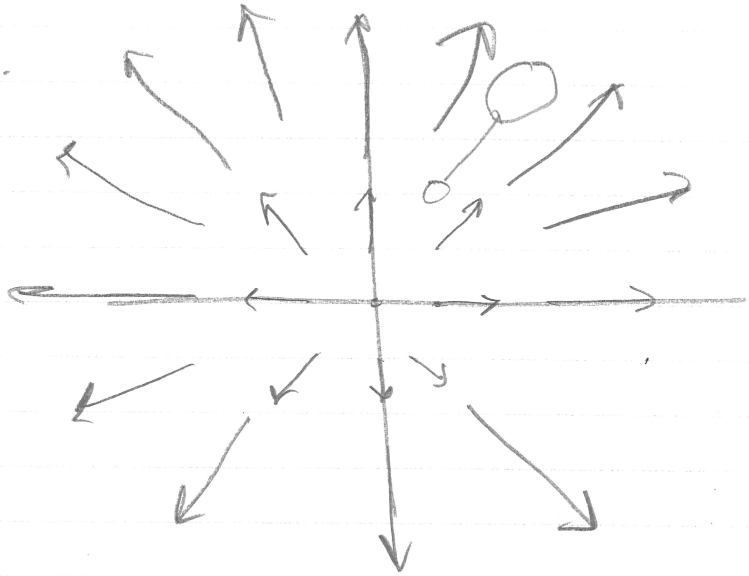


Div

Ex 4| Consider $\vec{F}(x,y) = x\hat{i} + y\hat{j}$

$\text{div}(\vec{F}) = 1+1 = 2.$

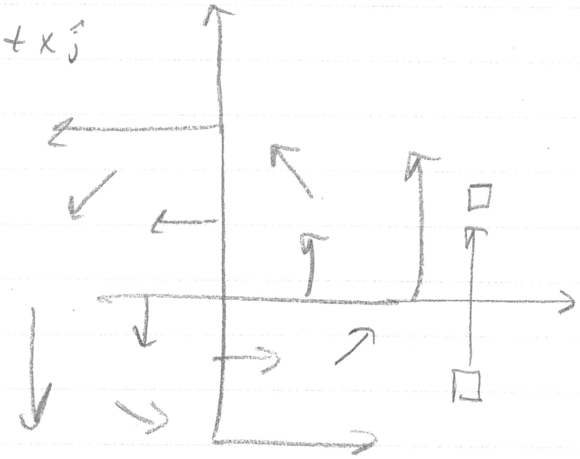
Fluid is expanding.



Ex 5| If $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$

$\text{div}(\vec{F}) = 0 + 0 = 0$

Fluid is just moving, it's incompressible.



Ex 6| What is the diver[g]ence of $\vec{F}(x,y,z) = \sin(yz)\hat{i} + x^3yz^2\hat{j} + (x+zy)\hat{k}$ at $(1,1,1)$?

$\nabla \cdot \vec{F} = 0 + x^3z^2 + y$

$\nabla \cdot \vec{F}(1,1,1) = 1^3 1^2 + 1 = 2.$

Curl

Defn: Curl (of a vector field)
If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \langle F_1, F_2, F_3 \rangle$, the curl of $\vec{F}$ is the vector field

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Only works in $\mathbb{R}^3$ or $\mathbb{R}^2$ w/ $z=0$

Ex 7) If $\vec{F}(x,y,z) = xz\hat{i} + xyz\hat{j} - y^2\hat{k}$, Find curl$(\vec{F})$.

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \hat{i}(-2y - xy) - \hat{j}(0 - x) + \hat{k}(yz - 0)$$

$$= (-2y - xy)\hat{i} + x\hat{j} + yz\hat{k}$$

Ex 8) What is the curl of $\vec{F}(x,y,z) = -zK$?

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & -z \end{vmatrix}$$

$$= \hat{i}(0-0) - j(0-0) + \hat{k}(0-0) = \vec{0}$$

What does curl represent physically?
This is harder to discuss. It's related to rotation.
If we put a stick in the fluid, will it rotate around?
Yes, if $\nabla \times \vec{F} \neq \vec{0}$
No, if $\nabla \times \vec{F} = \vec{0}$, it just flows in the water. Then $\vec{F}$ is irrotational.

Defn: Scalar Curl ($n=2$)
If $\vec{F}(x,y) = F_1\hat{i} + F_2\hat{j}$, then

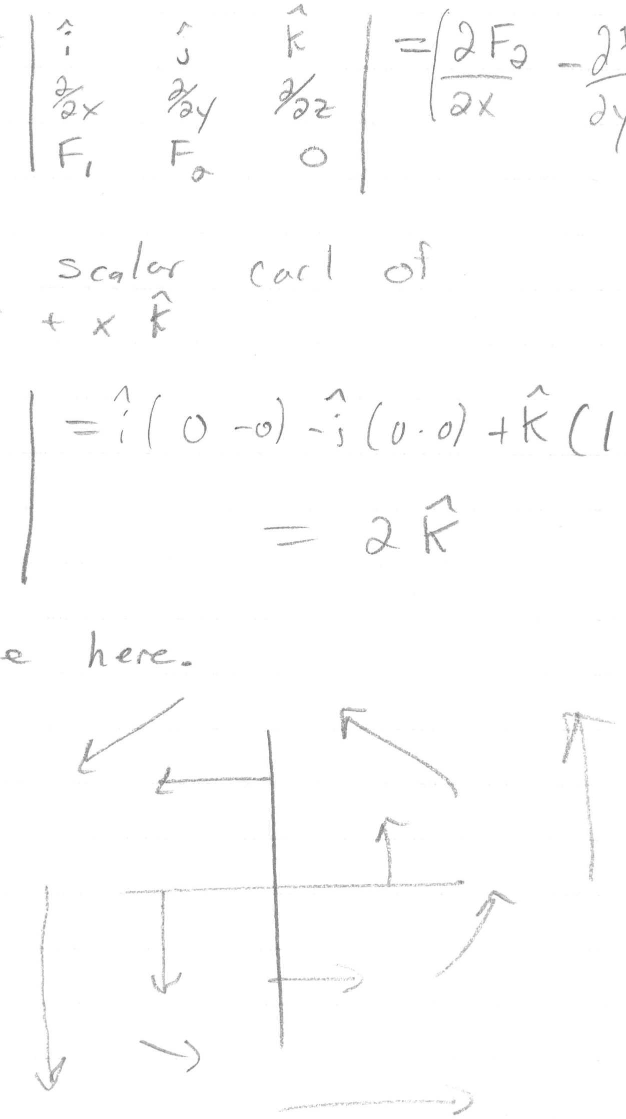
$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \hat{k}$$

Ex 9] What is the scalar curl of
$\vec{F}(x,y) = -y\hat{i} + x\hat{k}$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0 \cdot 0) + \hat{k}(1 - -1)$$

$$= 2\hat{k}$$

A stick will rotate here.



Ex 10) Show that $\vec{F}(x,y,z) = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$ is irrotational. ($\nabla \times \vec{F} = \vec{0}$)

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= \hat{i}(6xyz^2 - 6xyz^2) - \hat{j}(3y^2z^2 - 3y^2z^2) + \hat{k}(2yz^3 - 2yz^3)$$

$$= \vec{0}.$$

How do ∇f, $\text{div}(\nabla f)$, $\text{curl}(\nabla f)$ work together?

Thm 1: Gradients are curl free.
For any C^2 function f,

$$\text{curl}(\nabla f) = \boxed{\nabla \times \nabla f = \vec{0}.}$$

proof. $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

But f is C^2, so the mixed partials are equal, so

$$\nabla \times \nabla f = \vec{0}.$$

Thm: curl$(\vec{F}) = \vec{0} \Rightarrow$ Conservative $\vec{F}$

If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions are C^1 and curl$(\vec{F}) = \vec{0}$, then $\vec{F}$ is a conservative vector field.

Ex 101 cont. $\vec{F}(x,y,z) = y^2z^3\,\hat{i} + 2xyz^3\,\hat{j} + 3xy^2z^2\,\hat{k}$

Since curl$(\vec{F}) = \vec{0}$

$\vec{F}$ is conservative, so there exists f, s.t. $\nabla f = \vec{F}$. What's f?

$$= \langle \underset{f_x}{y^2z^3}, \underset{f_y}{2xyz^3}, \underset{f_z}{3xy^2z^2} \rangle$$

$$f(x,y,z) = \int f_x \, dx = xy^2z^3 + g(y,z)$$

$$f_y = 2xyz^3 + \frac{\partial g}{\partial y} = 2xyz^3$$

$$\text{So} \quad \frac{\partial g}{\partial y} = 0 \quad \Rightarrow \quad g(y,z) = h(z)$$

$$f_z(x,y,z) = 3xy^2z^2 + \frac{dh}{dz} = 3xy^2z^2$$

$$\frac{dh}{dz} = 0 \Rightarrow h(z) = C$$

So, $f(x,y,z) = xy^2z^3 + C$ is f.

Ex 9) cont. $\vec{F}(x,y) = -y\,\hat{i} + x\,\hat{j}$

since $\text{curl}(\vec{F}) = 2\hat{k}$

$\vec{F}$ is not conservative.

Thm 2: Curls are divergence free.

For any C^2 vector field $\vec{F}$, then

$$\text{div}(\text{curl}(\vec{F})) = \nabla \cdot (\nabla \times \vec{F}) = 0.$$

proof. Recall

$$\text{curl}(\vec{F}) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

So, $\nabla \cdot \text{curl}(\vec{F}) =$

$$\frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \underbrace{\frac{\partial^2 F_3}{\partial x\,\partial y}} - \utilde{\frac{\partial^2 F_2}{\partial x\,\partial z}} + \underline{\underline{\frac{\partial^2 F_1}{\partial y\,\partial z}}} - \underbrace{\frac{\partial^2 F_3}{\partial y\,\partial x}} + \utilde{\frac{\partial^2 F_2}{\partial z\,\partial x}} - \underline{\underline{\frac{\partial^2 F_1}{\partial z\,\partial y}}}$$

$= 0.$ By Clairaut's Thm

Ex 11 Show that the vector field

$$\vec{F}(x,y,z) = xz\,\hat{i} + xyz\,\hat{j} - y^2\hat{k}$$

can't be the curl of some vector field $\vec{F}$, that is $\text{curl}(\vec{G}) \neq \vec{F}$ for some $\vec{G}$.

If this were true then, $\text{div}(\text{curl}(\vec{G})) = 0$

But $\text{div}(\vec{F}) = z + xz \neq 0$.

Thus, $\vec{F}$ cannot be the curl of another vector field.

Defn: Laplacian (Laplace Operator)

The Laplace Operator, which operates on f, is

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Ex 12 Let $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} \left(= \frac{1}{r}\right)$ $(x,y,z) \neq (0,0,0)$

$$\frac{\partial f}{\partial x} = \frac{-1}{2}\,\frac{2x}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{dz} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-1(x^2+y^2+z^2)^{3/2} - x\,\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x}{(x^2+y^2+z^2)^3} = \frac{-(x^2+y^2+z^2) + 3x^2(1)}{(x^2+y^2+z^2)^{5/2}}$$

$$\nabla^2 f = \frac{-3(x^2+y^2+z^2) + 3x^2 + 3y^2 + 3z^2}{(x^2+y^2+z^2)^{5/2}} = 0.$$

Properties, ∇f, curl, div

properties of Derivatives

1. $\nabla(f+g) = \nabla f + \nabla g$

2. $\nabla(cf) = c\nabla f$, c-constant

3. $\nabla(fg) = g\,\nabla(f) + f\,\nabla(g)$

4. $\nabla\left(\frac{f}{g}\right) = [g\nabla f - f\,\nabla(g)]/g^2$, where $g(\vec{x}) \neq 0$.

distribution of •, ×

5. $\mathrm{div}(\vec{F}+\vec{G}) = \mathrm{div}\,\vec{F} + \mathrm{div}\,\vec{G}$

6. $\mathrm{curl}(\vec{F}+\vec{G}) = \mathrm{curl}(\vec{F}) + \mathrm{curl}(\vec{G})$

product rule [7. $\mathrm{div}(f\vec{F}) = \mathrm{div}(\vec{F}) + \vec{F}\cdot\nabla f$

Defn 8. $\mathrm{div}(\vec{F}\times\vec{G}) = \vec{G}\cdot\mathrm{curl}(\vec{F}) - \vec{F}\cdot\mathrm{curl}(\vec{G})$

Thm 2 { 9. $\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$

P.R. 10. $\mathrm{curl}(f\vec{F}) = f\,\mathrm{curl}(\vec{F}) + \nabla f \times \vec{F}$

Thm 1 { 11. $\mathrm{curl}(\nabla f) = \vec{0}$

#3 twice → 12. $\nabla^2(fg) = f\,\nabla^2 g + g\,\nabla^2 f + 2(\nabla f\cdot\nabla g)$

8 + 11 ⇒ 13. $\mathrm{div}(\nabla f \times \nabla g) = 0$

1,7 ⇒ 14. $\mathrm{div}(f\nabla g - g\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$